Principal series representations of $GL_2(F)$

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Disclaimer This notes have been made in preparation for a one hour talk about principal series representations of $\mathrm{GL}_2(F)$ in a study group about automorphic representations. The content is based on [1] and [2]. They have been made for my own benefit and they can contain mistakes.

1 Important subgroups of GL_2

Notation Let F be a non-archimedean local field and denote by $\mathcal{O}$ and $\mathfrak{p}$ its ring of integers and its maximal ideal, respectively. Denote by κ the residue field of F and by q its cardinality. Also, fix a generator ϖ of $\mathfrak{p}$.

We will consider the group $G = \mathrm{GL}_2(F)$ of non-singular 2-dimensional matricies with entries in F and the product of matrices as the operation. The following subgroups of F will be important ($*$ means any element in F)

- Maximal compact subgroup $K_0 = \mathrm{GL}_2(\mathcal{O})$.
- Borel subgroup $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$
- Unipotent subgroup $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$
- Diagonal torus $T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

Remark 1. $B = T \ltimes N$

Proposition 2. ([1, section 7]) The group G satisfies the following decompositions.

- Bruhat decomposition
$$G = B \cup BwN$$
where $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- Iwasawa decomposition
$$G = BK_0$$

- Cartan decomposition

$$G = \bigsqcup_{a \leq b \in \mathbb{Z}} K_0 \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K_0$$

Proposition 3. The group G is unimodular.

Proof. Since the Haar measure of a compact group is left and right invariant, then $\delta_G(k) = 1$ for every k contained in any compact subgroup of G, like K_0. Consider the matrix

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$$

Note that all the matrices appearing in the Cartan decomposition are of the form $g^b w g^a w$. Since $w \in K_0$, then $\delta_G(w) = 1$ and we just have to prove that $\delta_G(g) = 1$. Consider the Iwahori subgroups

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 : c \in \mathfrak{p} \right\}, \quad \overline{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 : b \in \mathfrak{p} \right\}$$

Since they have the same index in K_0, they also have the same Haar measure. Hence

$$\mu(I) = \mu(\overline{I}) = \mu(gIg^{-1}) = \delta_G(g)\mu(I) \Rightarrow \delta_G(g) = 1$$

□

Proposition 4. The modulus character of B is given by

$$\delta_B : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \frac{|d|}{|a|}$$

Proof. Since every element of N is contained in an open compact subgroup, then $\delta_B(n) = 1 \ \forall n \in N$.

Consider the matrix g from the proof of proposition 3 and let $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \cap K_0$. Since $g^{-1}hg = \begin{pmatrix} a & \varpi b \\ 0 & d \end{pmatrix}$, $g^{-1}Bg$ has index q in B. Thus $\delta_B(g) = q^{-1}$.

Since the modulus character is trivial in N and in the centre of B, we deduce that

$$\delta_B \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right] = \frac{|d|}{|a|}$$

□

2 Representations of G

The goal of this talk is to clasify (some of) the irreducible representations of G. Those we will classify are the principal series representations, i.e., those that arise as a subrepresentation of an induced one from B.

Definition 5. Let V be a representation of G and let θ be a character of N. We denote

$$V_\theta := \frac{V}{\langle (n-\theta(n))v : \ n \in N \rangle}$$

When θ is the trivial character, the quotient

$$V_N := \frac{V}{(N-1)V}$$

is called the *Jaquet module* of V.

Proposition 6. ([1, section 8.1, lemma]) The functor $V \to V_N$ is exact and additive. Furthermore, some $v \in V$ belongs to the kernel of this map if and only if there is a compact subset $N_0 \subset N$ such that

$$\int_{N_0} nv \, d\mu_N = 0$$

Definition 7. A representation of G is called *supercuspidal* if $V_N \neq 0$.

The goal of this talk is to provide a classification theorem for the representations of G that are not supercusptidal. They are also called *principal series representations.*

Theorem 8. An irreducible, smooth representation V of G is not supercuspidal if and only if it is isomorphic to a G-subspace of $\mathrm{Ind}_B^G \chi$, for some character χ of T.

Proof. Assume that V is isomorphic to a G-subspace of $\mathrm{Ind}_B^G(\chi)$. Then

$$\mathrm{Hom}_T(V_N, \chi) \cong \mathrm{Hom}_B(V, \chi) \cong \mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \neq 0$$

where the first isomorphism comes from the fact that χ is trivial of N (as a representation of B), so any B-homomorphism $V \to \chi$ factors through V_N. The second isomorphism is due to Frobenius reciprocity.

The last group of homomorphism is not zero because $V \subset \mathrm{Ind}_B^G \chi$. Indeed, by Schur's lemma

$$\mathbb{C} \cong \mathrm{Hom}_G(V, V) \subset \mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \tag{1}$$

In particular, we get that $V_N \neq 0$, so V is supercuspidal.

Conversely, assume that $V_N \neq 0$ and choose some $v \in V \setminus \{0\}$. Since V is irreducible, then $V = Gv$. Let $K \subset K_0$ be a compact open subgroup fixing v. Since K has finite index in K_0, then $K_0 v$ is finitely generated. Since $G = BK_0$, then $K_0 v$ generates V as a B-representation and thus its image generate V_N over T.

Therefore, V_N is finitely generated over T. Choose a minimal generating set $\{u_1, \ldots, u_t\}$. By Zorn's lemma, V_N has a maximal T-subspace U containing $u_1, \ldots, u_{t-1}$ such that $u_t \notin U$. Then $\chi = V_N/U$ is an irreducible representation of T, hence a character.

Consider the surjection

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \cong \mathrm{Hom}_T(V_N, \chi) \twoheadrightarrow \mathrm{Hom}_T(\chi, \chi) \cong \mathbb{C}$$

Hence $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G \chi) \neq 0$ and, since V is an irreducible G-representation, then V is isomorphic to a subspace of $\mathrm{Ind}_B^G \chi$. □

3 Irreducibility of $\mathrm{Ind}_B^G \chi$

By theorem 8, we are interested in studying the G-subspaces of $\mathrm{Ind}_B^G \chi$, where χ is a character of T. Turns out that this representations are in most cases irreducible.

Theorem 9. Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and set $X = \mathrm{Ind}_B^G \chi$. Then

1. X is reducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character or $x \mapsto |x|^2$.
2. When X is reducible, it satisfies the following
   (a) The G-composition length of X is 2.
   (b) One composition factor of X has dimension 1 and the other is infinite dimensional.
   (c) X has a 1-dimensional subspace when $\chi_1\chi_2^{-1} = 1$ and a 1-dimensional quotient when $\chi_1\chi_2^{-1}(x) = |x|^2$.

The goal of this section will be to present a proof of theorem 9. However, the first step is to study X as a B representation. In order to do that, its Jaquet module has an expclicit structure due to the following lemma.

Lemma 10. (Restriction-Induction) Let U be a (non-necessarily irreducible) smooth representation of T and let $X := \mathrm{Ind}_B^G U$. There is a short exact sequence of representations of T

$$0 \longrightarrow U^w \otimes \delta_B^{-1} \longrightarrow X_N \longrightarrow U \longrightarrow 0$$

where δ_B is the modular character of B and U^w is the conjugate representation U in which $b \in B$ acts by the action of wbw^{-1} on U.

Proof. There is a surjective B-map $X \to U$ given by $f \mapsto f(1)$. Call V the kernel of this map, so there is an exact sequence of representations of B

$$0 \longrightarrow V \longrightarrow X \longrightarrow U \longrightarrow 0$$

By proposition 6, the following is also exact

$$0 \longrightarrow V_N \longrightarrow X_N \longrightarrow U \longrightarrow 0$$

To complete the proof, we need to show that $V_N = U^w \otimes \delta_B^{-1}$.

Note that $f \in V$ if and only if $f(b) = 0 \; \forall b \in B$. Since $G = B \cup BwN$, this is equivalent to $\mathrm{supp}(f) \subset BwN$. For every $x \in F$, consider the matrix $g_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. When $|x|$ is small enough, the smoothness of f implies that $f(g_x) = 0$. Considering the identity

$$g_x = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} w \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \in Bw \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

we see that $\mathrm{supp}(f) \subset BwN_0$, for some open compact $N_0 \subset N$.

For every $f \in V$, define a function

$$f_N : \ T \to U, \ x \mapsto \int_N f(xwn)\, dn$$

By the definition of the modular character and the left invariance of the Haas measure,

$$(tf)_N(x) = \int_N f(xwnt)\, dn = \delta_B^{-1}(t) \int_N f(xt^w wn)\, dn = \delta_B^{-1}(t)(t^w f_N)(x)$$

Hence we have a B-homomorphism

$$V \to \delta_B^{-1} \otimes U^w : \ f \mapsto f_N(1)$$

which induces an isomorphism $V_N \cong \delta_B^{-1} \otimes U^w$ by proposition 6. □

Proposition 11. As a B-representation, $\mathrm{Ind}_B^G \chi$ has composition length 3. Two composition factors have dimension one, so the third one is infinite dimensional.

Proof. By lemma 10, there is an exact sequence of B-representations

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}_B^G \chi \longrightarrow \mathbb{C} \longrightarrow 0$$

where $V_N \cong \delta_B^{-1}\chi^w$. In particular, V_N is one dimensional. Denote by $V(N)$ the kernel of the map $V \to V_N$. We have an exact sequence

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

Hence we just need to prove that $V(N)$ is irreducible.

Claim: For all $\theta \in \hat{N}$, V_θ is one dimensional.

We have already proven the claim for the trivial character 1_N, since V_N is isomprhic to the character $\delta_B^{-1}\chi^w$.

To extend this result for other characters θ, note that $f \in X$ belongs to V if and only if its support is contained in BwN. Hence there is an N-equivariant isomorphism from

$$\Psi : \ V \to C_c^\infty(N), \ f \mapsto (n \mapsto f(wn))$$

In this new setting, we have an isomorphism

$$C_c^\infty(N)_N \to C_c^\infty(N)_\theta, \ f(n) \mapsto \theta(n)\mathrm{f}(n)$$

Hence V_θ is one dimensional.

Now we can prove that $V(N)$ is irreducible as a B-representation. Let W be a non-zero subrepresentation. Since N is abelian, all its irreducible representations are characters. Hence there is some non-trivial character θ such that $W_\theta \neq 0$. Given a different non-trivial character θ', there is some $x \in F$ such that $\theta'(y) = \theta(xy)$ for every $y \in F$. Denoting by

$$m_x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Then the action of m_x induces an isomorphism $W_\theta \to W_{\theta'}$, since $m_y \in B$ and W is a B-representation. Hence $W'_\theta \neq 0$. Hence $(V/W)_\theta = 0$ for all characters of N, so $W = V$. □

We will now study $\operatorname{Ind}_B^G \chi$ as a G-representation.

Proposition 12. Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Then $X = \operatorname{Ind}_B^G \chi$ contians a (unique) one-dimensional G-subspace if and only if $\chi_1 = \chi_2$.

Proof. Assume f spans a G-stable subspace. Then $f \notin V$ since $\operatorname{supp}(f)$ has to be right G-invariant and it cannot be contained in BwN_0 for some compact $N_0 \subset N$, as it happens with all the functions in V (see the proof of lemma 10).

The canonical N-map $X \to \mathbb{C} \cong X/V$ identifies the N-space $\mathbb{C}f$. Hence $nf = f \; \forall n \in N$. Take $x \in F$ and consider the identity

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}$$

Since f is locally constant, then f is fixed under right tranlsation by $\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}$ when $|x|$ is sufficiently large. Since f is also fixed by N, we have

$$f(w) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1)$$

for all $x \in F^*$ of sufficiently large absolute value. Thus $\chi_1 = \chi_2$.

Conversely, assume that $\chi_1 = \chi_2$. Then, as a B-representation, then

$$\chi(b) = \chi_1(\det b) \; \forall b \in B$$

Thus

$$X = \chi_1(\det g) \operatorname{Ind}_B^G(\mathbb{C}_B) \tag{2}$$

where $\mathbb{C}_B$ is the trivial B-representation. The function

$$f: \; G \to \mathbb{C}: \; g \mapsto \chi_1(\det g)$$

generates an invariant subspace. □

Proof of theorem 9. Assume X is reducible. By proposition 11, X contains a finite dimensional subspace or a finite dimensional quotient.

Assume the first alternative. Since V does not contain any finite dimensional G-subspace, X contains a one dimensonsal G-subspace L satisfying that $L \cap V = 0$. Thus we are in the case when $\chi_1 = \chi_2$.

The quotient X/L is thus isomporphic to V, which has B-composition length 2 and a unique one dimensional quotient $V_N \cong (\chi_1 \circ \det)\delta_B^{-1}$. If it was a G quotient, then G would act on V_N by a $\phi \circ \det$, where ϕ is a character of F^*. Indeed, since the commutator subgroup of $\mathrm{GL}_2(F)$ is $\mathrm{SL}_2(F)$, then all characters of G are of this form. Hovever, this is not possible by proposition 4.

In the case when X has a finite dimensional quotient, then $X^\vee$ has a one dimensional subspace. By the duality theorem,

$$X^\vee = \operatorname{Ind}_B^G \delta_B^{-1}\chi^{-1}$$

Hence we are in the situation where $\chi_1\chi_2^{-1}(x) = |x|^2$. □

4 Steinberg representation and final classification of non-cuspidal representations.

Definition 13. Let $\chi_1, \chi_2 : F^* \to \mathbb{C}^*$ be two smooth characters and denote by $\chi = \chi_1 \otimes \chi_2$ the corresponding character of T. Consider the representation

$$I(\chi_1, \chi_2) = \operatorname{Ind}_B^G(\delta_B^{-1/2} \otimes \chi)$$

Explicitly, this representation is described as

$$I(\chi_1, \chi_2) = \left\{ f : G \to \mathbb{C} \text{ locally constant} : f\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right] = \sqrt{\frac{|a|}{|d|}} \chi_1(a)\chi_2(d) f(g) \right\}$$

By the duality theorem, we have the following result

Proposition 14.

$$I(\chi) = I(\chi_1, \chi_2)^\vee \cong I(\chi_1^{-1}, \chi_2^{-1})$$

Definition 15. The *Steinberg representation* of G, denoted by St_G is defined by the exact sequence

$$0 \longrightarrow \mathbb{C}_G \longrightarrow \operatorname{Ind}_B^G(\mathbb{C}_B) \longrightarrow \mathrm{St}_G \longrightarrow 0$$

where $\mathbb{C}_G$ and $\mathbb{C}_B$ denote the trivial representations of G and B, respectively.

Before proving the self-duality of the Steinberg representation, consider the folllowing lemma.

Lemma 16. Let χ and ξ be characters of T. Then the space $\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_G^B \xi)$ has dimension 1 if $\xi = \chi$ or $\chi = \chi^w \delta_B^{-1}$.

Proof. By Frobenius reciprocity,

$$\operatorname{Hom}_G(\operatorname{Ind}_B^G \chi, \operatorname{Ind}_G^B \xi) \cong \operatorname{Hom}_T((\operatorname{Ind}_B^G \chi)_N, \xi)$$

By lemma 10, the Jaquet module fits as a T-representation into the exact sequence

$$0 \longrightarrow \chi^w \delta_B^{-1} \longrightarrow (\operatorname{Ind}_B^G \chi)_N \longrightarrow \chi \longrightarrow 0$$

If $\chi \neq \chi^w \delta_B^{-1}$, the above exact sequence splits and the lemma is clear. When $\chi = \chi^w \delta_B^{-1}$ then $\operatorname{Ind}_B^G \chi$ is irreducible and the lemma follows by Schur's lemma. $\square$

Proposition 17. The Steinberg representation is self-dual.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathbb{C}_G \longrightarrow \operatorname{Ind}_B^G \mathbb{C}_B \longrightarrow \mathrm{St}_G \longrightarrow 0$$

Dualising it,

$$0 \longrightarrow St_G^\vee \longrightarrow \operatorname{Ind}_B^G \delta_B^{-1} \longrightarrow \mathbb{C}_G \longrightarrow 0$$

By lemma 16, we have that

$$\operatorname{Hom}(\operatorname{Ind}_B^G \mathbb{C}_B, \operatorname{Ind}_B^G \delta_B^{-1}) \cong \mathbb{C}$$

Since those representations are not isomorphic, any non-zero homomorphism induces an isomorphism $\mathrm{St}_G \cong \mathrm{St}_G^\vee$. $\square$

Considering all the above, the final classification theorem follows.

Theorem 18. The isomorphism classes of irreducible, non-cuspidal representations of G are

- $I(\chi_1, \chi_2)$, where $\chi_1 \chi_2^{-1} \neq |\cdot|^{\pm 1}$.
- The one dimensional representations $\phi \circ \det$, where ϕ ranges over the characters of F^*.
- The special representations $(\phi \circ \det) \operatorname{St}_G$, where ϕ ranges over the characters of F^*.

The classes in this list are all distinct except $I(\chi_1, \chi_2) \cong I(\chi_2, \chi_1)$ in the first case.

References

[1] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for GL(2)*. Vol. 335. Springer Verlag, 2006.

[2] David Loeffler. "Modular Forms and Representatiosns of GL(2)". In: (2018).