Global Kirillov models

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Disclaimer This notes have been made in preparation for a one hour talk about principal series representations of $\mathrm{GL}_2(F)$ in a study group about automorphic representations. The content is based on [1]. They have been made for my own benefit and they may contain mistakes.

1 Global Kirillov models

Recall that $S_{k,t}$ was the space of adelic modular forms of weight (k,t). Let $\mathcal{F} \in S_{k,t}$ and, for every $x \in \mathbb{A}_f$, define $\gamma_x := \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\mathcal{F}_x = \mathcal{F}(\gamma_x, \tau)$$

We have seen that $\mathcal{F}_x$ is a modular form for some level Γ, so it has a Fourier expansion. Indeed, since $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some $h \in \mathbb{Z}$, $\mathcal{F}_x$ admits a Fourier expansion in $e^{2\pi i n/h}$ for $n \in \mathbb{Z}$. That motivates the following definition.

Definition 1. Let $\mathcal{F} \in S_{k,t}$ and let $a_1(\mathcal{F}_x)$ be the Fourier coefficient associated to $e^{2\pi i \tau}$. Define

$$\phi_{\mathcal{F}}(x) := a_1(\mathcal{F}_x)$$

Proposition 2. (Properties of $\phi_{\mathcal{F}}$)

1. $\phi_{\mathcal{F}}$ is a smooth function on $\mathbb{A}_\mathrm{f}^\times$.
2. There exists an open compact subset $W \subset \mathbb{A}_\mathrm{f}$ such that $\phi_{\mathcal{F}}$ is supported in $W \cap \mathbb{A}_\mathrm{f}^\times$.
3. Let $q \in \mathbb{Q}_+^\times$. Then

   $$\phi_{\mathcal{F}}(qx) = q^{-t} a_q(\mathcal{F}_x)$$

   Proof. By the automorphic form relation, $\mathcal{F}(\gamma_n \gamma_x, \tau) = q^{-t}\mathcal{F}(\gamma_x, q\tau)$. Hence $a_1(\mathcal{F}_{qx}) = q^{-t} a_q(\mathcal{F}_x)$. ☐

4. The association $\mathcal{F} \to \phi_{\mathcal{F}}$ is injective.

   Proof. By (iii), $\phi_{\mathcal{F}}$ determines $\mathcal{F}_x(\tau)$ for all $x \in \mathbb{A}_\mathrm{f}^\times$ and $\tau \in \mathcal{H}$. It determines $\mathcal{F}$ because a there is a set of representatives fo $\mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathrm{f})/U$ contained in $\begin{pmatrix} \mathbb{A}_\mathrm{f}^\times & 0 \\ 0 & 1 \end{pmatrix}$. ☐

5. Let $\theta: \mathbb{A}_f^\times \to \mathbb{C}^\times$ be the unique smooth character satisfying that $\Theta(x) = e^{-2\pi i x}$ forall $x \in \mathbb{Q}$. Then

$$\phi_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\mathcal{F}}(x) = \theta(bx)\phi_{\mathcal{F}(ax)}$$

Our goal is to study the space $S_{k,t}$.

Definition 3. A *cuspidal automorphic representation (CAR)* π is an irreducible subrepresentation of $S_{k,t}$.

Proposition 4. We have

$$S_{k,t} = \bigoplus_{\pi \text{ CAR of wt. } (k,t)} \pi$$

Definition 5. The *global Kirillov model* of π is

$$K(\pi) = \{\phi_{\mathcal{F}} : \mathcal{F} \in \pi\}$$

2 Review of local Kirillow models

Let θ_ℓ the continuous extension to $\mathbb{Q}_\ell$ of the character θ defined in proposition 2 (v).

Recall the unipotent sugroup

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Q}_\ell)$$

Let π_ℓ be an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$.

Theorem 6. (Kirillov)

$$\dim_{\mathbb{C}} \mathrm{Hom}_N(V, \theta) = 1$$

Remark 7. Kirillov's theorem holds true for every non-trivial character θ.

Definition 8. Let $\lambda \in \mathrm{Hom}_N(V, \theta) \setminus \{0\}$. For $v \in \pi_\ell$, the *Kirillov function* is

$$\phi_v : \mathbb{Q}_\ell^\times \to \mathbb{C}, \ a \mapsto \lambda \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v \right]$$

The *Kirillov model* of π_ℓ is

$$K(\pi_\ell, \theta_\ell) = \{\phi_v : v \in \pi_\ell\}$$

Kirillov models were used to prove the following theorem:

Theorem 9. (Casselman) Assume π_ℓ is not 1-dimensional. Let

$$U_n := \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_\ell) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \ell^n \right\}$$

1. There exists $n \in \mathbb{N}$ such that $V^{U_n} \neq 0$.
2. Let the conductor c of π_ℓ be the minimal n such that $V^{U_n} \neq 0$. If $m \geq c$, then

$$\dim \left(V^{U_m}\right) = m - c + 1$$

3 Relation between the local and global Kirillov models

The relation between adelic and local representations is given by restricted tensor products:

Definition 10. Suppose we have a collection of vector spaces X_ℓ, with ℓ being a prime, and vectors $x_\ell^0 \in X_\ell$, non-trivial for almost all ℓ. Define the *restricted tensor product*

$$\bigotimes_v{}' (X_\ell, x_\ell^0) \subset \bigotimes_\ell X_\ell$$

as the subspace spanned by tensors $\otimes x_\ell$ satisfying that $x_\ell = x_\ell^0$ for almost all ℓ.

This definition is motivated by the following theorem:

Theorem 11. (Flath's tensor product theorem) Let π be an irreducible smooth representationn of $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$. Then there are uniquely determined irreducible representations π_ℓ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ and $\phi_\ell^0 \in \pi_\ell^{\mathrm{GL}_2(\mathcal{O}_\ell)}$ such that

$$\pi \cong \bigoplus_\ell{}' (\pi_\ell, \phi_\ell^o)$$

Remark 12. In Flath's tensor product theorem, almost all π_ℓ are spherical, as they admit a non-zero $\mathrm{GL}_2(\mathbb{Z}_\ell)$-fixed vector, i.e., almost all π_ℓ can be seen as simple $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell), e_{GL_2(\mathbb{Z}_\ell)})$-modules. Moreover, it implies that $c(\pi_\ell) = 0$ for almost all ℓ.

Example 13. If we endow $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ with the subspace topology of the injection

$$\mathrm{GL}_2(\mathbb{A}_\mathrm{f}) \hookrightarrow \mathbb{A}_\mathrm{f}^4 \times \mathbb{A}_\mathrm{f}^\times,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d, \det(\gamma)^{-1})$$

Then the adelic Hecke algebra can be expressed as a restricted tensor product of the local Hecke algebras:

$$\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_\mathrm{f})) = \bigotimes_\ell{}' (\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell)), e_{GL_2(\mathbb{Z}_\ell)})$$

where $\mathcal{H}(G)$ is the set of compactly supported complex functions on G with the convolution product and $e_{\mathrm{GL}_2(\mathbb{Z}_\ell)}$ is the normalised characteristic function of $\mathrm{GL}_2(\mathbb{Z}_\ell)$.

Proposition 14. Let π be a CAR and let $\pi \cong \bigotimes_\ell' \pi_\ell$ be the isomorphism given by Flath's tensor product theorem. Then π_ℓ is an infinite-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ and we have an isomorphism

$$K(\pi) \to \bigotimes_\ell{}' K(\pi_\ell, \theta_\ell),\ \phi_{\otimes x_\ell} \mapsto \otimes_\ell \phi_{x_\ell}$$

Corollary 15. (Multiplicity one) Let π_1, π_2 be two CAR of weight (k, t) such that $\pi_1 \cong \pi_2$ as $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$-representation. Then $\pi_1 = \pi_2$.

Proof. If $\pi_1 \cong \pi_2$, then $\pi_{1,\ell} \cong \pi_{2,\ell}$ by Flath's theorem. By uniqueness of local Kirillov models, $K(\pi_{1,\ell}, \theta_\ell) = K(\pi_{2,\ell}, \theta_\ell)$ as function spaces on $\mathbb{Q}_\ell^*$. By proposition 14, $K(\pi_1) = K(\pi_2)$. By proposition 2 (iv), the automorphic form $\mathcal{F}$ is recovered from its image $\phi_{\mathcal{F}} \in K(\pi_i)$. Hence $\pi_1 = \pi_2$. □

Corollary 16. (Strong multiplicity one) Let π_1, π_2 be cuspidal automorphic representations of weight (k,t). Suppose $\pi_{1,\ell} \cong \pi_{2,\ell}$ for almost all ℓ. Then $\pi_1 = \pi_2$.

Remark 17. The analogue of multiplicity one for SL_2 is true, but the naive analogue of strong multiplicity one is false.

4 Relation between CAR and newforms

Definition 18. Let ϕ be a CAR. Define the *conductor* of π as

$$c(\pi) = \prod_\ell \ell^{c(\pi_\ell)}$$

Remark 19. The conductor is well defined since almost all π_ℓ are spherical.

Remark 20. Recall that

$$U_1(N) = \left\{ \gamma \in \mathrm{GL}_2(\mathbb{A}_\mathrm{f}) : \ \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

If $c(\pi) \nmid N$, Casselman's theorem implies that $\pi^{U_1(N)} = 0$. Otherwise,

$$\dim_\mathbb{C} \pi^{U_1(N)} = \prod_\ell \ell^{v_\ell(N) - v_\ell(c(\pi))} = \#\text{divisors of } N/c(\pi)$$

Proposition 21. There is a bijection between the CARs of weight (k,t) and the normalised new eigenforms in $S_k(\Gamma_1(N))$ for some N. It sends a CAR π to the (normalised) modular form spanning $\pi^{U_1(c(\pi))}$.

Proof. (Sketch) Given a cuspidal representation π, let $f_\pi = \mathcal{F}_\pi(1, \cdot)$ for a generator $\mathcal{F}_\pi$ of $\pi^{U_1(c(\pi))}$.

Any Hecke operator T_ℓ or U_ℓ restricts to π because of corollary 15. Since the Hecke operator is

$$\left[U_1(N) \begin{pmatrix} \omega_\ell & 0 \\ 0 & 1 \end{pmatrix} U_1(N) \right]$$

it has also remain $\pi^{U_1(c(\pi))}$ invariant. Since $\pi^{U_1(c(\pi))}$ is one-dimensional, it has to act on $\mathcal{F}_\pi$ by multiplication by scalars.

We may rescale f_π such that $a_1(f_\pi) = 1$. Then f_π is a newform since π is orthogonal to other CAR with the scalar product in L^2. □

Corollary 22. The space of classical cusps forms descomposes as

$$S_k(\Gamma_1(N)) \cong \bigoplus_{c(\pi)|N} S_k(\Gamma_1(N))[\pi]$$

where

$$S_k(\Gamma_1(N))[\pi] = \left\langle f_\pi(d\tau) : \ d \mid \frac{N}{c(\pi)} \right\rangle$$

Corollary 23. If f_π is the nweform corresponding to π and $\pi \cong \otimes' \pi_\ell$. Then

$$\pi_\ell \cong I(\alpha_\ell, \beta_\ell) = \mathrm{Ind}_B^G(\alpha_\ell \otimes \beta_\ell)$$

where B is the Borel subgroup of $G = \mathrm{GL}_2(\mathbb{Q}_\ell)$ and α_ℓ, β_ℓ are two unramified characters ending ℓ to the roots of

$$X^2 - \frac{a_\ell(\mathcal{F}_\pi)}{\ell^{t-1/2}} + \ell^{k-2t}\varepsilon(\ell)$$

where ε is the nebentype character of f_π.