Bianchi modular forms

The aim of this study group is to introduce and study Bianchi modular forms. These objects – modular forms for GL_2 over an imaginary quadratic field – are receiving increasing attention from the number theory community, owing to a rich array of interesting research questions allied to recent breakthrough progresses arising from the work of Peter Scholze and many others.[1]

The topics we will consider will include:

- What a Bianchi modular form is, and why they are a natural generalisation of classical modular forms;
- The L-functions of classical and Bianchi modular forms, and (stating) the Bianchi modularity theorem;
- Modular symbols and their connections to L-functions, in both the classical and Bianchi cases;
- A flavour of p-adic topics.

Context and motivation. Let $E : y^2 = x^3 + ax + b$ be a rational elliptic curve, with $a, b \in \mathbf{Q}$. The *L-function* of E is defined (roughly) to be

$$L(E, s) \sim \prod_{\text{primes } p} (1 - a_p p^{-s} + p^{1-2s}), \qquad s \in \mathbf{C}, \qquad a_p = p + 1 - |E(\mathbf{F}_p)|$$

(ignoring primes of bad reduction). It converges absolutely to a holomorphic function in the right-half plane $\mathrm{Re}(s) > 3/2$. The *Birch–Swinnerton-Dyer (BSD) conjecture* says that

$$\mathrm{ord}_{s=1} L(E, s) = \mathrm{rank}(E(\mathbf{Q})),$$

where $E(\mathbf{Q}) = \mathbf{Z}^{\mathrm{rank}(E(\mathbf{Q}))} \times [\text{finite group}]$. It also precisely predicts the value of the leading term in terms of the Tate–Shafarevich group. Famously, when it was made, Tate remarkaed that the BSD conjecture...

> *"... relates the behavior of a function $L(E, s)$, at a point $(s = 1)$ where it is not at present known to be defined, to the order of a group which is not known to be finite."*

In particular, *a priori* we do not know that $L(E, s)$ can be continued from the right-half plane $\mathrm{Re}(s) > 3/2$ (where it is known to exist) to the value $s = 1$! This is a major, and difficult, problem.

At this point, modular forms enter the picture. Consider a cuspidal modular form $f(q) = \sum_{n\geqslant 1} a_n n^{-s}$ of weight k. To such a modular form one can attach an L-function, namely

$$L(f, s) = \sum_{n\geqslant 1} a_n n^{-s}.$$

[1] As a flavour: Scholze's tour-de-force work attaching Galois representations to torsion classes (Annals of Math, 2013) paved the way for decisive progress on decades-old open problems. In particular, it allowed the execution of a strategy suggested by Calegari–Geraghty to attack modularity lifting theorems in this setting. Notable milestones include:

- In 2018, Allen–Calegari–Caraiani–Gee–Helm–Le Hung–Newton–Scholze–Taylor–Thorne proved that Bianchi modular forms are potentially modular;
- Late last year (2022) Caraiani–Newton upgraded this to full modularity;
- Just last week (September 2023), Boxer–Calegari–Gee–Newton–Thorne proved the Ramanujan and Sato–Tate conjectures for Bianchi modular forms, by proving 'potential' existence of symmetric power lifts.

These pioneering works will be beyond the scope of this study group, but by the end we should be able to *state* the theorems they proved.

This converges absolutely in the right-half plane $\mathrm{Re}(s) > (k+1)/2$. Now, crucially, one can express $L(f, s)$ via an integral formula; for example, we have $L(f, s) = \int_0^\infty f(iy)y^{s-1}dy$. Using this one can prove that $L(f, s)$ admits analytic continuation to $\mathbf{C}$, and satisfies a functional equation relating $L(f, s)$ and $L(f, k - s)$.

The *Shimura–Taniyama conjecture*, now known as the *modularity theorem*, says there exists a modular form f_E of weight 2 such that

$$L(f_E, s) = L(E, s).$$

It was proved in many cases by Wiles, and formed the key step in his proof of Fermat's last theorem. Note an immediate major consquence: $L(E, s)$ admits analytic continuation to all of $\mathbf{C}$, and in particular the value $s = 1$; and it satisfies a functional equation relating $L(E, s)$ and $L(E, 2 - s)$, making $s = 1$ the 'centre of symmetry'. To this day, no proof of these facts is known that doesn't go through the modularity theorem.

Beyond its applications to FLT and deducing properties of L-functions, the modularity theorem is a prototype case for the Langlands programme, which predicts a much wider relationship between 'arithmetic objects' (such as elliptic curves) and 'automorphic objects' (such as modular forms). A natural question arises: what if we consider more general elliptic curves? For example, what does Langlands predict if we allow a and b to be elements of some number field F, rather than just in $\mathbf{Q}$?

The major focus of this study group will be when F is an imaginary quadratic field, e.g. $\mathbf{Q}(i)$. For example, consider the elliptic curve E defined by the (long) Weierstrass equation

$$y^2 + (i + 1)\, xy + iy = x^3 + (239i - 399)\, x - 2869i + 2627.$$

This has conductor norm 65, and admits no model over $\mathbf{Q}$ (so is not isomorphic to a rational elliptic curve). The Langlands programme predicts that the correct 'automorphic' object to pair with this curve is a *Bianchi modular form* f_E. This was a major open problem for decades, but was proved last year by Caraiani–Newton. As a result of their recent work, we now know analytic continuation and functional equations of the L-functions of elliptic curves over imaginary quadratic fields.

In this study group, we will introduce Bianchi modular forms, and explain why they are natural objects in number theory, and why they should correspond to elliptic curves over imaginary quadratic fields. This will involve studying L-functions over number fields. We will also see modular symbols in this setting, algebraic analogues ripe for computations.

References for the study group (obviously non-exhaustively) include:

[**Wei71**] Weil's 1971 book on automorphic forms.

[**Bump**] Bump's canonical textbook on automorphic forms.

[**Wil16**] My 2016 PhD thesis 'Overconvergent modular symbols over number fields'. Part I contains a summary of Weil's book.

Talks: schedule and descriptions

Talk 1: Overview. ???

This talk will introduce the overarching framework in which Bianchi modular forms sit, and preview some of the topics we'll study.

Talk 2: The adeles and Dirichlet characters as automorphic forms. ???

In this talk, we will recap the adeles, and describe how Dirichlet characters – that is, characters of $(\mathbf{Z}/N\mathbf{Z})^\times$ for some N – can be thought of as adelic automorphic forms for the group GL_1. This provides important motivation for the more advanced setting of classical modular forms.

References: I'm sure we can find a decent reference for adeles.

For the characters theory: Chapter I of my lecture notes 'An introduction to p-adic L-functions II: modular forms'.

Talks 3 and 4: Classical modular forms via adeles. ???

In these two talks, we will explain how classical modular forms – holomorphic functions on the upper half-plane satisfying suitable symmetry conditions – can be thought of as adelic automorphic forms on GL_2. This switch of language was arguably one of the most important developments in 20th century number theory, paving the way for the Langlands programme and the introduction of Representation Theory as a major tool in the study of modular forms.

In the first talk, we'll discuss passing from classical modular forms to functions of $\mathrm{GL}_2(\mathbf{R})$, and then to functions on $\mathrm{GL}_2(\mathbf{A})$. This will only focus on the automorphy conditions, the adelic version of the condition

$$f(\gamma z) = (cz+d)^k f(z).$$

In the second talk, we'll discuss the generalisations of being holomorphic (harmonicity) and bounded at the cusps (B-moderacy). This will involve some differential geometry. This is included for completeness and won't be used again later in the study group (so perhaps we should omit it?)

References: Weil's book or Chapters 2 and 3 of my thesis (for the first and second talk respectively).

Talk 5: Bianchi modular forms on the adeles. ???

In this talk, we'll discuss how to generalise adelic automorphic forms from $\mathrm{GL}_2(\mathbf{A_Q})$ to $\mathrm{GL}_2(\mathbf{A}_F)$, where F is an imaginary quadratic field.

References: Weil or Chapter 2.2 of my thesis

Talk 6: Bianchi modular forms classically. ???

In this talk, we will pass from the adelic description of Bianchi modular forms to the more classical description, as vector-valued forms on the upper half-space. It should also include the Fourier expansion (the generalisation of q-expansions).

References:

Talk 7: L-functions. This talk will introduce the L-functions of classical and Bianchi modular forms, and discuss integral formulas for their value at $s = 1$. It can also discuss modularity in this setting (viewed via L-functions; i.e. needs to also discuss the L-function of an elliptic curve).

Talk 8: Modular symbols (classically). ???

This talk will define classical modular symbols in the style of Pollack–Stevens, and discuss their connections to L-values.

References: Pollack–Stevens: their ENS paper, or their AWS notes.
My lecture notes on p-adic L-functions for modular forms.

Talk 9: Modular symbols (Bianchi). ???

This talk will define Bianchi modular symbols and discuss connections to L-values.

References: My thesis.

Talk 10: Base-change. ???

Not so clear what to have here, without referring to automorphic representations. Perhaps I can fold this into the next talk.

Talk 11: p-adic stuff, or cohomology?. ???

I can give a seminar-style talk on some p-adic topics? Or discuss work of Venkatesh et. al on dimensions of cohomology groups in the Bianchi setting? There's a lot of very rich and interesting stuff that could go here.

References: The literature.