

and Fitting ideals o Selmer

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Kolyvagin systems and Fitting ideals of Selmer group of rank 0

Alberto Angurel Andrés

University of Nottingham

30/09/2025



General picture

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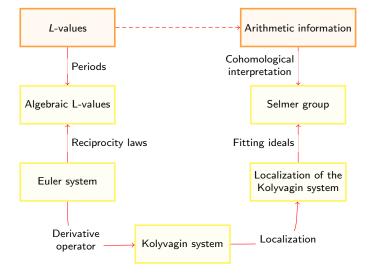
Kolyva system

Structur of Selme groups

Euler

Elliptic Curves

Proofs





■ We are going to focus on

Localization Kolyvagin system Group structure of $\kappa_n \in H^1(\mathbb{Q}, T)$ $\mathrm{loc}_p(\kappa_n) \in H^1(\mathbb{Q}_p, T)$ the Selmer group

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Kolyvagin system $\kappa_n \in H^1(\mathbb{Q},T) \xrightarrow{\text{Localization}} \text{Group structure of the Selmer group}$

- We cannot apply the theory of Kolyvagin systems directly, because
 - The classical Selmer group is self-dual, so its core rank is zero.
 - There are no non-zero Kolyvagin systems for this Selmer group.

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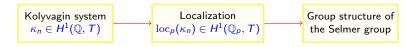
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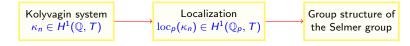
Duality

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Elliptic Curves Proofs We are going to focus on



- We cannot apply the theory of Kolyvagin systems directly, because
 - The classical Selmer group is self-dual, so its core rank is zero.
 - There are no non-zero Kolyvagin systems for this Selmer group.
- The general theory of Kolyvagin systems only describes the structure of the Selmer group *restricted at p*.
- We extend this theory to Selmer groups of rank zero by considering Kolyvagin systems over an auxiliary Selmer structure.



Setting and assumptions

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■ (H0) Let $p \ge 5$ and let **T** be free \mathbb{Z}_p -module of finite rank endowed with a continuous action of $G_{\mathbb{Q}}$, ramifying only at a finite amount of primes.

• (H1) $\rho: G_{\mathbb{Q}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.

■ (H2) (will appear later)



Selmer groups are formed by the elements of the global cohomology groups $H^1(\mathbb{Q}, \mathbf{T})$ that satisfy *local conditions*.



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lacktriangle What is a local condition? A local condition for a prime ℓ is a choice of a subgroup

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})\subset H^1(\mathbb{Q}_\ell,\mathsf{T})$$

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Definition (Selmer pre-structure)

A Selmer pre-structure \mathcal{F} is a choice of a local condition for every prime (including the archimedean one).



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Definition (Selmer pre-structure)

A Selmer pre-structure \mathcal{F} is a choice of a local condition for every prime (including the archimedean one).

Definition (Selmer group)

The Selmer group for \mathcal{F} is defined as

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathbf{T}):=\ker\left(H^1(\mathbb{Q},\mathbf{T})\to\bigoplus_{\ell}\frac{H^1(\mathbb{Q}_{\ell},\mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_{\ell},\mathbf{T})}\right)$$

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Definition (finite cohomology)

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}):=\mathsf{ker}\left(H^1(\mathbb{Q},\mathsf{T})\to H^1(I_\ell,\mathsf{T}\otimes\mathbb{Q}_p)\right)$$



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Definition (Selmer structure)

A Selmer structure is a Selmer pre-structure such that there is a finite set of primes Σ such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathsf{T})=H^1_f(\mathbb{Q}_\ell,\mathsf{T})\;orall \ell
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$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) \; \forall \ell \notin \Sigma$$

Proposition (Selmer groups)

If \mathbb{Q}_{Σ} denotes the maximal extension of \mathbb{Q} unramified outside Σ , we have that

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},\mathbf{T}) = \ker \left(H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q},\mathbf{T}) \to \prod_{\ell \in \Sigma} \frac{H^1(\mathbb{Q}_{\ell},\mathbf{T})}{H^1_{\mathcal{F}}(\mathbb{Q}_{\ell},\mathbf{T})} \right)$$

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Corollary

 $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathsf{T})\subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathsf{T})$ is a finitely generated \mathbb{Z}_p -module.

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Definition (Fitting ideal)

Fitting ideals

Let M be a finitely generated R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

 $\operatorname{Fitt}_{i}^{R}(M)$ is the ideal generated by the minors of size (m-i) of A.

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Fact: Fitting ideals are well defined.

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Let R be a DVR (with maximal ideal $\mathfrak m$ and residue field κ). Then

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

for some non-negative integers $r, s, \alpha_1 \geq \ldots \geq \alpha_s$.



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Proposition

$$i \in \{0,\ldots,r-1\} \Rightarrow \operatorname{Fitt}_i(M) = (0)$$

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- $i \ge r + s \Rightarrow \operatorname{Fitt}_i(M) = (1).$

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The Fitting ideals determine i up to isomorphism:

Proofs



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The Fitting ideals determine *i* up to isomorphism:

r is the minimum i such that $Fitt_i(M) \neq 0$.

Alberto Angurel Andres



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Corollary

The Fitting ideals determine i up to isomorphism:

- r is the minimum i such that $Fitt_i(M) \neq 0$.
- For i > 0, $\alpha_i = \text{Fitt}_{r+i+1}(M)\text{Fitt}_{r+i}(M)^{-1}$.



Definition (dual Galois modules)

- Pontryagin dual: $\mathbf{T}^{\vee} = \operatorname{Hom}(\mathbf{T}, \mathbb{Q}_p/\mathbb{Z}_p)$.
- Cartier dual: $\mathbf{T}^* = \operatorname{Hom}(\mathbf{T}, \mu_{p^{\infty}}).$



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Proposition (local duality)

The cup-product induces a non-degenerate pairing

$$H^1(\mathbb{Q}_\ell,\mathsf{T}) imes H^1(\mathbb{Q}_\ell,\mathsf{T}^*) o H^2(\mathbb{Q}_\ell,\mu_{\mathfrak{p}^\infty})\cong \mathbb{Q}_p/\mathbb{Z}_p$$

Moreover, $H_f^1(\mathbb{Q}_\ell, \mathbf{T})$ and $H_f^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ are exact annihilators of each other.

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Corollary

$$H^1(\mathbb{Q}_{\ell}, \mathbf{T})^{\vee} \cong H^1(\mathbb{Q}_{\ell}, \mathbf{T}^*)$$

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Corollary

$$H^1(\mathbb{Q}_\ell,\mathsf{T})^\vee\cong H^1(\mathbb{Q}_\ell,\mathsf{T}^*)$$

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Dual Selmer structure

Definition (dual Selmer structure)

The dual Selmer structure \mathcal{F}^* is defined by the local conditions

$$\mathit{H}^1_{\mathcal{F}^*}(\mathbb{Q}_\ell,\mathsf{T}^*):=\mathit{Ann}ig(\mathit{H}^1_\mathcal{F}(\mathbb{Q}_\ell,\mathsf{T})ig)\subset \mathit{H}^1(\mathbb{Q}_\ell,\mathsf{T}^*)$$

These are the elements of $H^1(\mathbb{Q}_\ell, \mathbf{T}^*)$ which annihilate $H^1_{\mathcal{T}}(\mathbb{Q}_\ell, T)$ under the local duality pairing.



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Remark (well defined)

The dual Selmer structure is well defined since

$$H^1_f(\mathbb{Q}_\ell,\mathsf{T}^*):=\mathrm{Ann}(H^1_f(\mathbb{Q}_\ell,\mathsf{T}))$$

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Global duality

Let ${\mathcal F}$ and ${\mathcal G}$ be Selmer structures such that

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)\subset H^1_{\mathcal{G}}(\mathbb{Q}_\ell,T)\ \forall \ell$$

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Then the dual local conditions satisfy the opposite relations

$$H^1_{\mathcal{G}^*}(\mathbb{Q}_\ell, T^*) \subset H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell, T^*) \ orall \ell$$



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Clearly,

$$\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathsf{T})\subset\mathrm{Sel}_{\mathcal{G}}(\mathbb{Q},\mathsf{T}),\quad \mathrm{Sel}_{\mathcal{G}^*}(\mathbb{Q},\mathsf{T}^*)\subset\mathrm{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathsf{T}^*)$$



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Global duality

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{G}}(\mathbb{Q}, T) \longrightarrow \prod_{\ell} \frac{H^{1}_{\mathcal{G}}(\mathbb{Q}_{\ell}, T)}{H^{1}_{\mathcal{F}}(\mathbb{Q}_{\ell}, T)} \longrightarrow \operatorname{Sel}_{\mathcal{F}^{*}}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$



Assumptions

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Proofs

■ (H0) Let $\mathfrak{p} \geq 5$ and let **T** be free \mathbb{Z}_p -module of finite rank endowed with a continuous action of $G_{\mathbb{Q}}$, ramifying only at a finite amount of primes.

• (H1) $\rho: G_{\mathbb{Q}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.

• (H2) $H^1(\mathbb{Q}_\ell, \mathbf{T})/H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathbf{T})$ is a torsion-free \mathbb{Z}_p -module.



SLIDE

Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi: \ \mathbf{T} \to T$ to the canonical projection.

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Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

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Under assumptions (H0), (H1) and (H2), the following equality holds true.

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Under assumptions (H0), (H1) and (H2), the following equality holds true.

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*) = \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}^*)[p^k]$$

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Proof:



Fix $k \in \mathbb{N}$ and let $T = \mathbf{T}/p^k$. Denote $\pi : \mathbf{T} \to T$ to the canonical projection.

Definition (propagated local condition)

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T)=\pi\left(H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbf{T})\right)$$

Proposition

Under assumptions (H0), (H1) and (H2), the following equality holds true.

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*) = \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}^*)[p^k]$$

Remark

A study of $Sel_{\mathcal{F}}(\mathbb{Q}, \mathbf{T}/p^k)$ for all k will determine $Sel_{\mathcal{F}}(\mathbb{Q}, \mathbf{T})$.

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Proof

Definition

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Definition

A prime ℓ is a Kolyvagin prime if

- $\ell \equiv 1 \mod p^k$.
- $P_{\ell}(1) = \det(1 \operatorname{Frob}_{\ell}|T) = 0.$

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Notation

P denotes the set of Kolyvagin primes.

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 $\mathcal{N}(\mathcal{P})$ denotes the set of square free products of Kolyvagin primes.

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Notation

P denotes the set of Kolyvagin primes.

 $\mathcal{N}(\mathcal{P})$ denotes the set of square free products of Kolyvagin primes.

 $\mathcal{N}_i(\mathcal{P})$ denotes the set of square free products of exactly i Kolyvagin primes.



Transverse local condition and finite-singular map

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Definition (transverse local condition)

$$H^1_{tr}(\mathbb{Q}_\ell, T) := \operatorname{Im} \left(H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell, T) \to H^1(\mathbb{Q}_\ell, T) \right)$$



Transverse local condition and finite-singular map

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$$H^1_{tr}(\mathbb{Q}_\ell,T):=\mathrm{Im}ig(H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell,T) o H^1(\mathbb{Q}_\ell,T)ig)$$

Proposition (split of the local cohomology group)

If ℓ is Kolyvagin prime, then

$$H^1(\mathbb{Q}_\ell,T)=H^1_f(\mathbb{Q}_\ell,T)\oplus H^1_{tr}(\mathbb{Q}_\ell,T)\cong \mathbb{Z}/p^k\oplus \mathbb{Z}/p^k$$

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Transverse local condition and finite-singular map

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If ℓ is Kolyvagin prime, then

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Definition (finite-singular map)

There is a canonical isomorphism

$$\phi_{fs}: H^1_f(\mathbb{Q}_\ell, T) \cong H^1_{tr}(\mathbb{Q}_\ell, T)$$

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Let $a, b, c \in \mathbb{N}$ be such that abc is square free.

Assume all primes dividing a, b and c are Kolyvagin primes.

We can define a new Selmer structure $\mathcal{F}_a^b(c)$ by

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Proofs



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$$H^1_{\mathcal{F}^b(c)}(\mathbb{Q}_\ell,T):=H^1_{\mathcal{F}}(\mathbb{Q}_\ell,T) \text{ if } \ell
mid abc.$$

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$$\blacksquare \ H^1_{\mathcal{F}^b_2(c)}(\mathbb{Q}_\ell,\,T)=0 \ \text{if} \ \ell\mid a.$$

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$$\blacksquare \ H^1_{\mathcal{F}^b_{\sigma}(c)}(\mathbb{Q}_\ell,T)=H^1(\mathbb{Q}_\ell,T) \text{ if } \ell\mid b.$$

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$$lacksquare H^1_{\mathcal{F}^b_{\mathcal{E}}(c)}(\mathbb{Q}_\ell,T)=H^1(\mathbb{Q}_\ell,T) \ ext{if} \ \ell\mid b.$$

•
$$H^1_{\mathcal{F}^b(c)}(\mathbb{Q}_\ell, T) = H^1_{tr}(\mathbb{Q}_\ell, T)$$
 if $\ell \mid c$.

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Definition (Kolyvagin system)

A Kolyvagin system is a collection of elements

$$\kappa_n \in \mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \subset H^1(\mathbb{Q}, T)$$

for every $n \in \mathcal{N}(\mathcal{P})$, satisfying the following Kolyvagin conditions.

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For every $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ such that $\ell \nmid n$, consider the localization maps at ℓ .

$$\operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathcal{F}(n)}(\mathbb{Q}_{\ell},T) = H^1_f(\mathbb{Q}_{\ell},T)$$

$$\operatorname{loc}_{\ell}(\kappa_{n\ell}) \in H^1_{\mathcal{F}(n\ell)}(\mathbb{Q},T) = H^1_{tr}(\mathbb{Q}_{\ell},T)$$

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The Kolyvagin condition for $n \in \mathcal{N}(\mathcal{P})$ and $\ell \in \mathcal{P}$ is

$$\phi_{fs}(\operatorname{loc}_{\ell}(\kappa_n)) = \operatorname{loc}_{\ell}(\kappa_{n\ell})$$

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Notation

The module of Kolyvagin systems will be denoted by $KS(T, \mathcal{F})$.

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Core rank

Definition/proposition (core rank)

There exists a non-negative integer $\chi(\mathcal{F})$ and a non-canonical homomorphism

$$\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q},T)\cong \left(\mathbb{Z}/p^k
ight)^{\chi(T)}\oplus \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},T^*)$$

(possibly after swapping the roles of T and T^* .)

The integer $\chi(T)$ is called the core rank of T.

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(possibly after swapping the roles of T and T^* .)

The integer $\chi(T)$ is called the core rank of T.

Proposition (Sakamoto, 2021)

$$\chi(\mathcal{F}_{\mathsf{a}}^{\mathsf{b}}(\mathsf{c})) = \chi(\mathcal{F}) + \nu(\mathsf{b}) - \nu(\mathsf{a})$$

where $\nu(b)$ and $\nu(a)$ are the number of primes dividing b and a, respectively.

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Core rank and Kolyvagin systems

Theorem (Mazur-Rubin, 2004)

• If $\chi(\mathcal{F}) = 0$, then $KS(\mathcal{T}, \mathcal{F}) = 0$.

There are no Kolyvagin system to control the Selmer group. We will see a posible solution later in the talk.

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• If $\chi(\mathcal{F}) = 1$, then $\mathrm{KS}(\mathcal{T}, \mathcal{F}) \cong \mathbb{Z}/p^k$.

A generator of $KS(T, \mathcal{F})$ is called a primitive Kolyvagin system. We will see next that they carry information to compute all the Fitting ideals of the Selmer group $Sel_{\mathcal{F}^*}(\mathbb{Q}, T^*)$.

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Core rank and Kolyvagin systems

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■ If $\chi(\mathcal{F}) > 1$, then $KS(\mathcal{T}, \mathcal{F})$ is too large.

In order to compute the Selmer group, [Mazur-Rubin, 2016] and [Burns-Sakamoto-Sano, 2025] modified the definition of Kolyvagin system in (biduals of) exterior powers of the Selmer groups.



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Definition (order of a Kolyvagin element)

$$\operatorname{ord}(\kappa_n) := \max \left\{ j \in \{0, \dots, k\} : \kappa_n \in p^j H^1_{\mathcal{F}(n)}(\mathbb{Q}, T) \right\}$$



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If κ is a primitive Kolyvagin system

$$\operatorname{ord}(\kappa_n) = \min \left\{ k, \operatorname{length}\left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)\right) \right\}$$



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Definition

$$\Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n): n \in \mathcal{N}_i\}}$$

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Theorem (Mazur-Rubin, 2004)

When $\chi(\mathcal{F})=1$ and κ is a primitive Kolyvagin system

$$\Theta_i = \operatorname{Fitt}_i \left(\operatorname{Sel}_{\mathcal{F}^*} (\mathbb{Q}, T)^* \right)$$

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Proof

Core rank 0

- We have seen that there are no non-zero Kolyvagin systems.
- Choose a prime ℓ such that $H^1_s(\mathbb{Q}_\ell,T)\cong \mathbb{Z}/p^k$.
- Note that all Kolyvagin primes satisfy the above condition, but we do not restrict to them.
- Then \mathcal{F}^{ℓ} is cartesian and $\chi(\mathcal{F}^{\ell})=1$.



Core rank 0

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Definition

Let $\kappa \in \mathrm{KS}(T, \mathcal{F}^{\ell})$. Define

$$\delta_n = \delta_n(\kappa) := \operatorname{loc}_{\ell}(\kappa_n) \in H^1_{\mathfrak{s}}(\mathbb{Q}_{\ell}, T) \cong \mathbb{Z}/p^k$$



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Definition (order)

$$\operatorname{ord}(\delta_n) = \max \left\{ j \in \{0, \dots, k\} : \delta^n \in \left(p^j\right) \right\}$$

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Core rank 0

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Proposition (Kim, 2025)

$$\operatorname{ord}(\delta_n) = \min \left\{ k, \operatorname{length} \left(H^1_{(\mathcal{F}^*)(n)}(\mathbb{Q}, \mathcal{T}^*) \right) \right\}$$



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$$\Theta_i := (p)^{\min\{\operatorname{ord}(\delta_n): n \in \mathcal{N}_i(\mathcal{P})\}} = \langle \{\delta_n: \ n \in \mathcal{N}_i(\mathcal{P})\} \rangle \subset \mathbb{Z}/p^k$$

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Theorem (A., 2025)

For all i, we have

$$\Theta_i \subset \operatorname{Fitt}_i \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right)$$

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Curves



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Theorem (A., 2025)

For all i, we have

$$\Theta_i\subset \mathrm{Fitt}_i\left(H^1_{\mathcal{F}^*}(\mathbb{Q},\,T^*)\right)$$

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$$\bullet \Theta_{i-1} \subsetneq \operatorname{Fitt}_{i-1} \left(H^1_{\mathcal{F}^*}(\mathbb{Q}, T^*) \right).$$

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Remark

When $T = \mathbf{T}/p^k$ for some \mathbb{Z}_{p} -module \mathbf{T} and k is large enough, the ideals Θ_i determine $H^1_T(\mathbb{Q}, \mathbf{T}^*)$ up to isomorphism



Galois representations which are not residually self-dual

Theorem (A., 2025)

Under the following assumption on non self-duality,

■ (N1) $T/p \ncong T^*[p]$.



Galois representations which are not residually self-dual

Theorem (A., 2025)

Under the following assumption on non self-duality,

$$\blacksquare (N1) \ T/p \ncong T^*[p].$$

Then for all $i \in \mathbb{Z}_{\geq 0}$, we have the equality

$$\Theta_i = \operatorname{Fitt}_i \left(H^1_{\mathcal{F}}(\mathbb{Q}, T^*) \right)$$

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Connection to Euler systems

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Assume that we have an Euler system z.

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Connection to Euler systems

- Assume that we have an Euler system z.
- The Kolyvagin derivative operator produces a Kolyvagin system for \mathbf{T}/p^k for all k and the canonical Selmer structure, defined as

$$\begin{cases} H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\ell,\mathsf{T}/p^k) = H^1_f(\mathbb{Q}_\ell,\mathsf{T}/p^k) \text{ if } \ell \neq p,\infty \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_p,\mathsf{T}/p^k) = H^1(\mathbb{Q}_p,\mathsf{T}/p^k) \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{R},\mathsf{T}/p^k) = H^1(\mathbb{R},\mathsf{T}/p^k) \end{cases}$$

This is also known as *relaxed at p*. Its dual Selmer structure will be called *restricted at p*.

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Connection to Euler systems

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$$\begin{cases} H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\ell,\mathsf{T}/\rho^k) = H^1_f(\mathbb{Q}_\ell,\mathsf{T}/\rho^k) \text{ if } \ell \neq \rho,\infty \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_\rho,\mathsf{T}/\rho^k) = H^1(\mathbb{Q}_\rho,\mathsf{T}/\rho^k) \\ H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{R},\mathsf{T}/\rho^k) = H^1(\mathbb{R},\mathsf{T}/\rho^k) \end{cases}$$

This is also known as *relaxed at p*. Its dual Selmer structure will be called *restricted at p*.

Theorem (Kolyvagin, 1995)

$$\operatorname{ord}(z_{\mathbb{Q}}) \geq \operatorname{ord}(\kappa_1) \geq \operatorname{length}\left(H^1_{(\mathcal{F}^{\operatorname{can}})^*}(\mathbb{Q},\mathsf{T}^*)\right)$$

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We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$.

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We apply the results to $\mathbf{T} = T_p \mathbf{E} \otimes \chi$. We assume the following:

- **(E0)** E is defined over \mathbb{Q} .
- \bullet (χ 1) The conductor of χ is not divisible by p or any bad prime of E.
- (χ^2) The order of χ is prime to p.

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Modularity There exists a modular form $f_{\chi} = \sum \chi(n) a_n q^n$ such that

$$T_{f_\chi} = T_p E \otimes \chi$$

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Kato constructed an Euler system for this representation.

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Elliptic curves: Bloch-Kato Selmer structure

Bloch-Kato Selmer structure

The classical local conditions are defined by Bloch-Kato condition

$$\begin{cases} H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_\ell, T) = H^1_f(\mathbb{Q}_\ell, T) & \forall \ell \neq p \\ H^1_{\mathcal{F}_{BK}}(\mathbb{Q}_p, T) = \ker \left(H^1(\mathbb{Q}_p, T) \to H^1(\mathbb{Q}_p, T \otimes_{Z_p} \mathcal{B}_{\operatorname{crys}}) \right) \end{cases}$$

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Elliptic curves: Bloch-Kato Selmer structure

Bloch-Kato Selmer structure

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Assume the following:

E1) $\rho: G_{\mathbb{O}} \to \operatorname{Aut}(\mathbf{T})$ is surjective.



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Proposition

Assuming (E1), \mathcal{F}_{BK} satisfies all the assumptions (H0), (H1) and (H2) and $\chi(\mathcal{F}_{BK})=0$.



Kolyvagin derivative

The Kolyvagin derivative operator applied to Kato's Euler system, produces a Kolyvagin system for $\mathcal{F}^{\operatorname{can}} = (\mathcal{F}_{BK})^p$.

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Kolyvagin derivative

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Denote K_{χ} to the fixed field of χ . Assume:

• $(\chi 3)$ $E((K_{\chi})_{\mathfrak{p}})[p] = \{O\}$ for every prime \mathfrak{p} above p.

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Proposition

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Proposition

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Assume further:

- $(\chi 4)$ The Tamagawa numbers of E over K_{χ} are prime to p.
- $(\chi 5)$ Iwasawa main conjecture (in the sense of Kato) holds for f_{χ} .

Proposition

The Kolyvagin derivative produced from Kato's Euler system is a primitive Kolyvagin system.



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Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \log_p(\kappa_n)$. Assume

(E2) The Manin constant is prime to p.



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Proposition (Kurihara numbers)

Let n be a square-free product of Kolyvagin systems for T/p^k .



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Let *n* be a square-free product of Kolyvagin systems for T/p^k .

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/nc)^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell \mid n} \log_{\eta_\ell}(a) \in \mathbb{Z}_p[\chi]/p^k$$



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lacksquare c is the conductor of χ .



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- $\left[\frac{a}{cn}\right]^{\pm}$ are the real and imaginary part of the modular symbols of E.
- η_{ℓ} is a primitive root of $(\mathbb{Z}/\ell)^{\times}$ and $\log_{\eta_{\ell}}(a)$ is the image of the logarithm under the projection $(\mathbb{Z}/\ell)^{\times} \cong \mathbb{Z}/(\ell-1) \twoheadrightarrow Z/p^k$.



Elliptic curves: Kurihara numbers

The final goal is to compute $\delta_{n,\chi} = \text{loc}_{p}(\kappa_{n})$. Assume

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Remark

If K/\mathbb{Q} is an abelian extension such that all the characters of $\mathrm{Gal}(K/\mathbb{Q})$ satisfy $(\chi 1)$ - $(\chi 5)$, then the twisted Kurihara numbers determine $\mathrm{Sel}(K, E[p^{\infty}])$ up to isomorphism of $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$ -modules.



ES

Proposition

Consider the exact sequence of (\mathbb{Z}/p^k) -modules

$$0 \longrightarrow C \longrightarrow M \xrightarrow{\phi} (\mathbb{Z}/p^k)^i$$

Then

$$(p)^{\operatorname{length}(\mathcal{C})} \subset \operatorname{Fitt}_i(M)$$

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If we choose ϕ and C maximizing the image of ϕ , the equality holds.

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Theorem (first inequality)

$$\Theta_i \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$$

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Proof Apply the proposition to

$$0 \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*) \longrightarrow \prod_{\ell \mid n} H^1_f(\mathbb{Q}_\ell, T^*) \cong (\mathbb{Z}/p^k)^{\nu(n)}$$

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Then

$$(p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_{(n)}(\mathbb{Q},\mathcal{T}^*))} \subset (p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_n(\mathbb{Q},\mathcal{T}^*))} \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},\mathcal{T}))$$



Proofs: first inequality

Proposition

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Then

$$(p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)(n)(\mathbb{Q},T^*))} \subset (p)^{\operatorname{length}(\operatorname{Sel}(\mathcal{F}^*)_n(\mathbb{Q},T^*))} \subset \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q},T))$$

The proof is completed by taking the minimum over all $n \in \mathcal{N}_i(\mathcal{P})$.

Alberto Angurel Andres



■ The localization map

$$\operatorname{loc}_{\ell}: \ \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_{\ell}, T^*)$$

has the largest possile image.

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This can be achieved using Chebotarev density theorem.

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■ We want to choose $n \in \mathcal{N}(\mathcal{P})$ such that

$$\operatorname{Sel}_{(\mathcal{F}^*)_n}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q}, T^*)$$

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Proposition

Assume the following localization map is surjective

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$$

Then

$$\operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) = \operatorname{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q}, T^*)$$

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Then

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Proof

$$0 \longrightarrow \operatorname{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \longrightarrow \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, T) \longrightarrow H^1_f(\mathbb{Q}_{\ell}, T)$$

$$\overbrace{\operatorname{Sel}_{(\mathcal{F}^*)^\ell(n)}}(\mathbb{Q},T^*)^\vee \longrightarrow \operatorname{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q},T^*)^\vee \longrightarrow 0$$

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$$\hookrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})^{\ell}(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow \operatorname{Sel}_{(\mathcal{F}^{*})(n)}(\mathbb{Q}, T^{*})^{\vee} \longrightarrow 0$$

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Proofs



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Proposition

$$\mathrm{Fitt}_{\mathit{i}}(\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},\mathit{T}^*)^{\vee}) = \Theta_{\mathit{i}} := (p)^{min\{\mathrm{ord}(\kappa_{\mathit{n}}): \mathit{n} \in \mathcal{N}_{\mathit{i}}(\mathcal{P})\}}$$



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Proposition

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee}) = \Theta_i := (p)^{\min\{\operatorname{ord}(\kappa_n) : n \in \mathcal{N}_i(\mathcal{P})\}}$$

Proof Inductively, assume we have constructed some n_i such that

$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

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$$\operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q}, T^*)^{\vee}) = p^{\operatorname{ord}(\kappa_{n_i})}$$

Since the core rank is one,

$$\mathrm{Sel}_{\mathcal{F}(n_i)}(\mathbb{Q},T)\cong (\mathbb{Z}/p^k)\oplus \mathrm{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q},T^*)$$

Then there exists a surjective map $\operatorname{Sel}_{\mathcal{F}(p_i)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.

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Proposition

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Proofs: equality in rank one

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By Chebotarev density theorem, we can find a prime ℓ_{i+1} such that

- $lacksquare{1}$ loc $_{\ell_{i+1}}$: Sel $_{\mathcal{F}(n_i)}(\mathbb{Q},T) \to H^1_f(\mathbb{Q}_\ell,T)$ is surjective.
- $\blacksquare \operatorname{loc}_{\ell_{i+1}} : \operatorname{Sel}_{\mathcal{F}^*(\eta_i)}(\mathbb{Q}, T^*) \to H^1_f(\mathbb{Q}_\ell, T^*)$ has maximal image.

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- $\blacksquare \operatorname{loc}_{\ell_{i+1}} : \operatorname{Sel}_{\mathcal{F}^*(n_i)}(\mathbb{Q}, T^*) \to H^1_{\mathfrak{f}}(\mathbb{Q}_{\ell}, T^*)$ has maximal image.

For $n_{i+1} := n_i \ell_{i+1}$, we get that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n_{i+1})}(\mathbb{Q},T^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell_{i+1}}(n_i)}(\mathbb{Q},T^*)$$

Moreover.

$$(p)^{\operatorname{ord}(\kappa_{n_{i+1}})} = (p)^{\operatorname{length}(\operatorname{Sel}_{(\mathcal{F}^*)(n_{i+1})})} = \operatorname{Fitt}_{i+1}(\operatorname{Sel}_{(\mathcal{F}^*)}(\mathbb{Q}, T))$$

Alberto Angurel Andres



Proofs: equality in rank zero: characteristic reduction

When $\chi(\mathcal{F}) = 0$,

$$\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T)\cong\mathrm{Sel}_{(\mathcal{F}^*)(n)}(\mathbb{Q},T)\ \forall n\in\mathcal{N}(\mathcal{P})$$

It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) \to \mathbb{Z}/p^k$.



Proofs: equality in rank zero: characteristic reduction

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It might not exist a surjective map $\mathrm{Sel}_{\mathcal{F}(n)}(\mathbb{Q},T) o \mathbb{Z}/p^k$.

By the structure theorem,

$$\mathrm{Sel}_{\mathcal{F}(n)} = \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $e_1 > \cdots > e_s$.

Trick Swap T by $T_{e_1} := T/p^{e_1}$.



Proofs: equality in rank zero: characteristic reduction

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Similarly, we can find a prime ℓ such that the maps

- \blacksquare Sel_{$\mathcal{F}(n)$}(\mathbb{Q}, T_{e_1}) $\to H^1_f(\mathbb{Q}_\ell, T_{e_1})$.
- $\blacksquare \operatorname{Sel}_{\mathcal{F}(n)}(\mathbb{Q}, (T_{e_1})^*) \to H^1_f(\mathbb{Q}_\ell, (T_{e_1})^*).$

are surjective. We obtain the following for the Selmer group over T_{e_1} .

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},(T_{e_1})^*) = \mathrm{Sel}_{(\mathcal{F}^*)_{\ell}(n)}(\mathbb{Q},(T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$



What information can we deduce from this to the Selmer group over \mathcal{T} ?

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What information can we deduce from this to the Selmer group over T?

$$\blacksquare \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, (T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$$

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What information can we deduce from this to the Selmer group over T?

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■ If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, \mathcal{T}^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

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■ When $e_1 = e_2$, we only know that

$$Sel_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset Sel_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong$$

$$\mathbb{Z}/p^k \times Sel_{\mathcal{F}_e(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_5}$$

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What information can we deduce from this to the Selmer group over T?

- $\blacksquare \ \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},\, T^*)[p^{e_1}] \cong \operatorname{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q},(\, T_{e_1})^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}.$
- If either $e_1 = k$ or $e_1 > e_2$, we can conclude that

$$\mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

■ When $e_1 = e_2$, we only know that

$$\begin{split} & \mathrm{Sel}_{(\mathcal{F}^*)(n\ell)}(\mathbb{Q}, T^*) \subset \mathrm{Sel}_{(\mathcal{F}^*)^{\ell}(n)}(\mathbb{Q}, T^*) \cong \\ & \mathbb{Z}/p^k \times \mathrm{Sel}_{\mathcal{F}_{\ell}(n)}(\mathbb{Q}, T) \cong \mathbb{Z}/p^k \times \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_5} \end{split}$$

■ The structure theorem implies that

$$\mathrm{Sel}_{(\mathcal{F}^*)(p\ell)}(\mathbb{Q}, \mathcal{T}^*) \cong \mathbb{Z}/p^{f_2} \times \mathbb{Z}/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$

for some $f_2 \geq e_2$.

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We start with $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},T)$ and choose a prime ℓ_1 such that the localization maps for T_{e_1} and $T_{e_1}^*$ are surjective, and minimizing f_2 .

We have two cases



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• If $f_2 = e_2$, then $\Theta_1 = \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.

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- If $f_2 = e_2$, then $\Theta_1 = \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.
- If $f_2 > e_2$, then $\Theta_1 \subsetneq \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$.

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- If $f_2 = e_2$, then $\Theta_1 = \text{Fitt}_1 \left(\text{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee} \right)$.
- If $f_2 > e_2$, then $\Theta_1 \subseteq \operatorname{Fitt}_1(\operatorname{Sel}_{\mathcal{F}^*}(\mathbb{Q}, T^*)^{\vee})$. In this case.

$$\mathrm{Sel}_{\mathcal{F}(\ell_1)} \cong \mathbb{Z}/p^{f_2} \times Z/p^{e_3} \times \cdots \times \mathbb{Z}/p^{e_s}$$

Since $f_2 > e_3$, we can choose a prime ℓ_2 in a way such that $f_3 = e_3$, so

$$\Theta_2 = \operatorname{Fitt}_2 \left(\operatorname{Sel}_{\mathcal{F}^*} (\mathbb{Q}, T^*)^{\vee} \right)$$



Non self-dual representations

Chebotarev density theorem is stronger in this case:

For every pair of subgroups $C \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T)$ and $D \subset \operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)$ such that the quotients $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T)/C$ and $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)/D$ are cyclic, we can find a prime ℓ such that the kernels of the localization maps are C and D.

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■ If $\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T) \cong \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_s}$, a technical argument constructs a prime ℓ such that

$$\mathrm{Sel}_{\mathcal{F}(\ell)}\cong \mathbb{Z}/p^{e_2} imes\cdots imes\mathbb{Z}/p^{e_5}$$

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• If $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q},T)\cong \mathbb{Z}/p^{e_1}\times \cdots \times \mathbb{Z}/p^{e_s}$, a technical argument constructs a prime ℓ such that

$$\mathrm{Sel}_{\mathcal{F}(\ell)} \cong \mathbb{Z}/p^{e_2} \times \cdots \times \mathbb{Z}/p^{e_s}$$

■ Therefore, the equality $\Theta_i = \operatorname{Fitt}_i(\operatorname{Sel}_{\mathcal{F}}(\mathbb{Q}, T^*)^{\vee})$ holds for all i.



