The method of Euler systems

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Class groups

Fundamental Theorem of Arithmetic

Let $\frac{a}{b} \in \mathbb{Q}^\times$. Then there is a **unique** set of (different) primes $p_1, \ldots, p_r$ and integers $\alpha_1, \ldots, \alpha_r$ such that

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$$(3) = \mathfrak{p} \cdot \overline{\mathfrak{p}}, \quad (7) = \mathfrak{q} \cdot \overline{\mathfrak{q}}, \quad (1 + 2\sqrt{-5}) = \overline{\mathfrak{p}} \cdot \mathfrak{q}, \quad (1 - 2\sqrt{-5}) = \mathfrak{p} \cdot \overline{\mathfrak{q}}$$

Class groups

Class group

The class group $Cl(K)$ is defined as the quotient

$$Cl(K) := \frac{\mathcal{I}_K}{\mathcal{P}_K}$$

where

- $\mathcal{I}_K$: group of (fractional) ideals of $\mathcal{O}_K$.
- $\mathcal{P}_K$: group of (fractional) principal ideals, i.e., those generated by a single element.

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Theorem (Dirichlet)

For every number field K, the class number h_K is finite, but its actual computation is, in general, a hard problem.

Fermat last theorem

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The equation

$$x^n + y^n = z^n$$

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It is enough to show the problem for n being

- $n = p$ odd prime.
- $n = 4$: follows from an analysis of the pythagorean triples.

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Indeed, note that the equation is equivalent to

$$\left(x^{n/p}\right)^p + \left(y^{n/p}\right)^p = \left(z^{n/p}\right)^p$$

If there are no solutions for $n = p$, there are no solutions for any multiple of p.

Fermat last theorem

Kummer's approach for $n = p \geq 5$

- Assume w.l.o.g. that x, y, z share no prime factor.
- Factorise the equation using the root of unity $\zeta_p := e^{\frac{2\pi i}{p}}$.

$$x^p + y^p = (x + y) \cdot (x + \zeta_p y) \cdots (x + (\zeta_p)^{p-1} y) = z^p$$

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- **If the field $\mathbb{Q}(\zeta_p)$ admits unique factorisation**, it follows that

$$x + \zeta_p y = u\alpha^p \text{ for some } \alpha \in \mathbb{Z}[\zeta_p], u \in \mathbb{Z}[\zeta_p]^\times$$

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- **Problem**: this argument only works when $h_p := h_{\mathbb{Q}(\zeta_p)} = 1$.

  But that does not hold all primes, being $p = 23$ the smallest prime with $h_p > 1$.

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Regular primes

The primes such that $p \nmid h_p$ are called regular primes. Smallest irregular primes are

$$p = 37, 59, 67, 101, 103, 131, \ldots$$

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General case (A. Wiles, 1995)

The proof uses the modularity theorem, which requires deeper mathematics.

Algebraic Curves

Rational points on algebrtaic curves

Let f be an homogeneous polynomial on 3 variables with coefficients in $\mathbb{Q}$.

$$C(\mathbb{Q}) := \{\mathbf{x} \in \mathbb{P}^2(\mathbb{Q}) : \ f(\mathbf{x}) = 0\}$$

We assume there are not singular points in C.

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Genus

- $g = 0 \rightarrow$ Conics: Either $C(\mathbb{Q}) = \emptyset$ or $C(\mathbb{Q}) \cong \mathbb{Q}$.
- $g = 1 \rightarrow$ Elliptic curves: our case of interest.
- $g > 1 \rightarrow C(\mathbb{Q})$ is finite (Faltings), but difficult to compute

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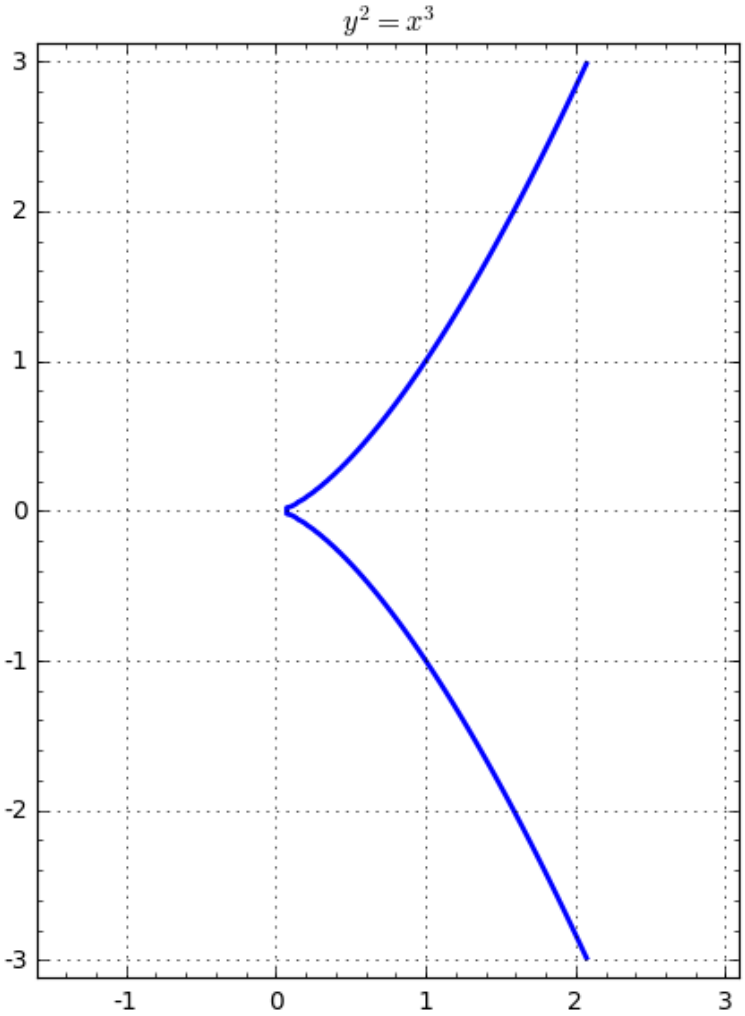
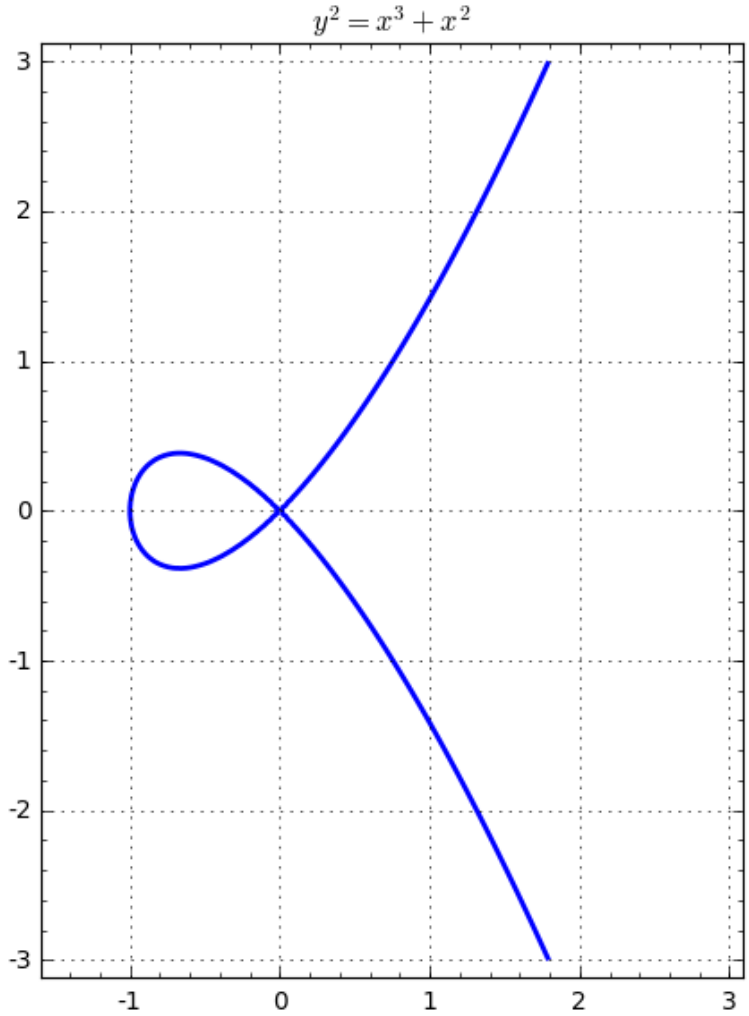
Elliptic curves

Using an algebraic transformation, we can assume that every elliptic curve is defined by an equation of the form

$$E : \ y^2 = x^3 + Ax + B, \ A, B \in \mathbb{Z}, \ 4a^3 + 27b^2 \neq 0$$

Singular elliptic curves





Local-global principle

Completions of the rationals

- The possible norms that can be defined on $\mathbb{Q}$ are the absolute value and the p-adic norms for every prime p.

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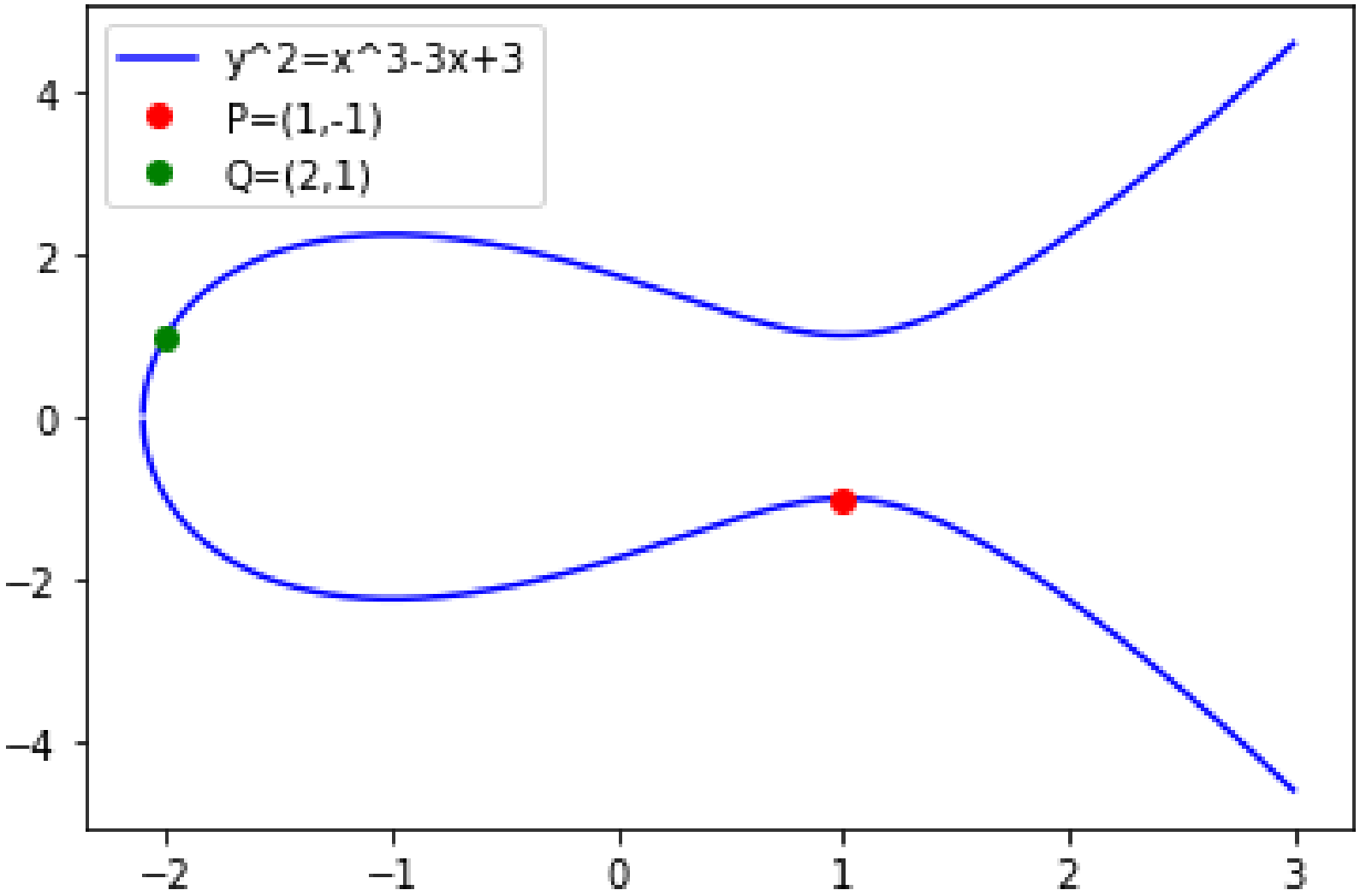
Elliptic curves

Hasse principle does not hold for elliptic curve. The failure of this principle is defined by the Tate-Shafarevich group.

Adding points on an elliptic curve



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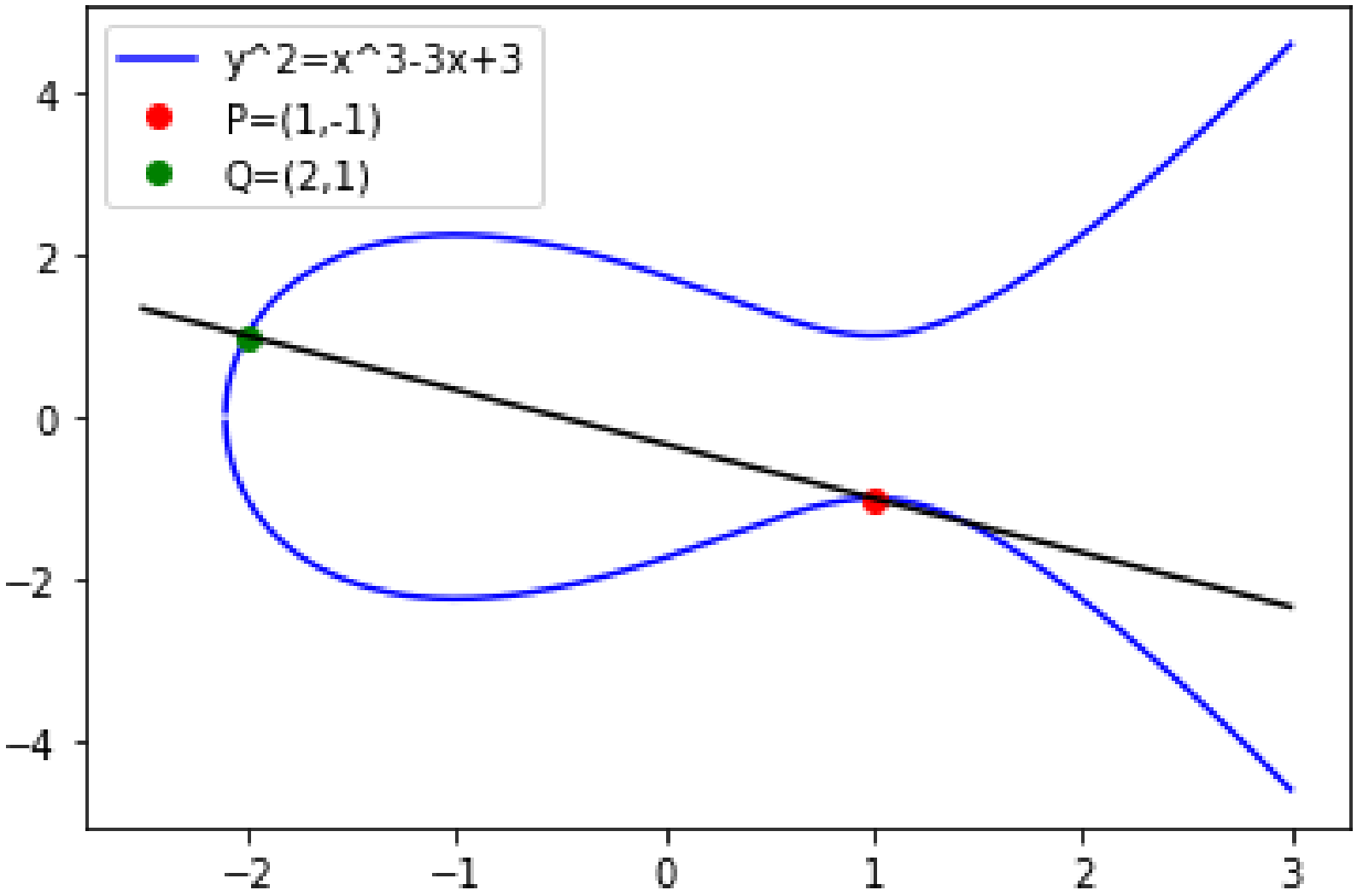

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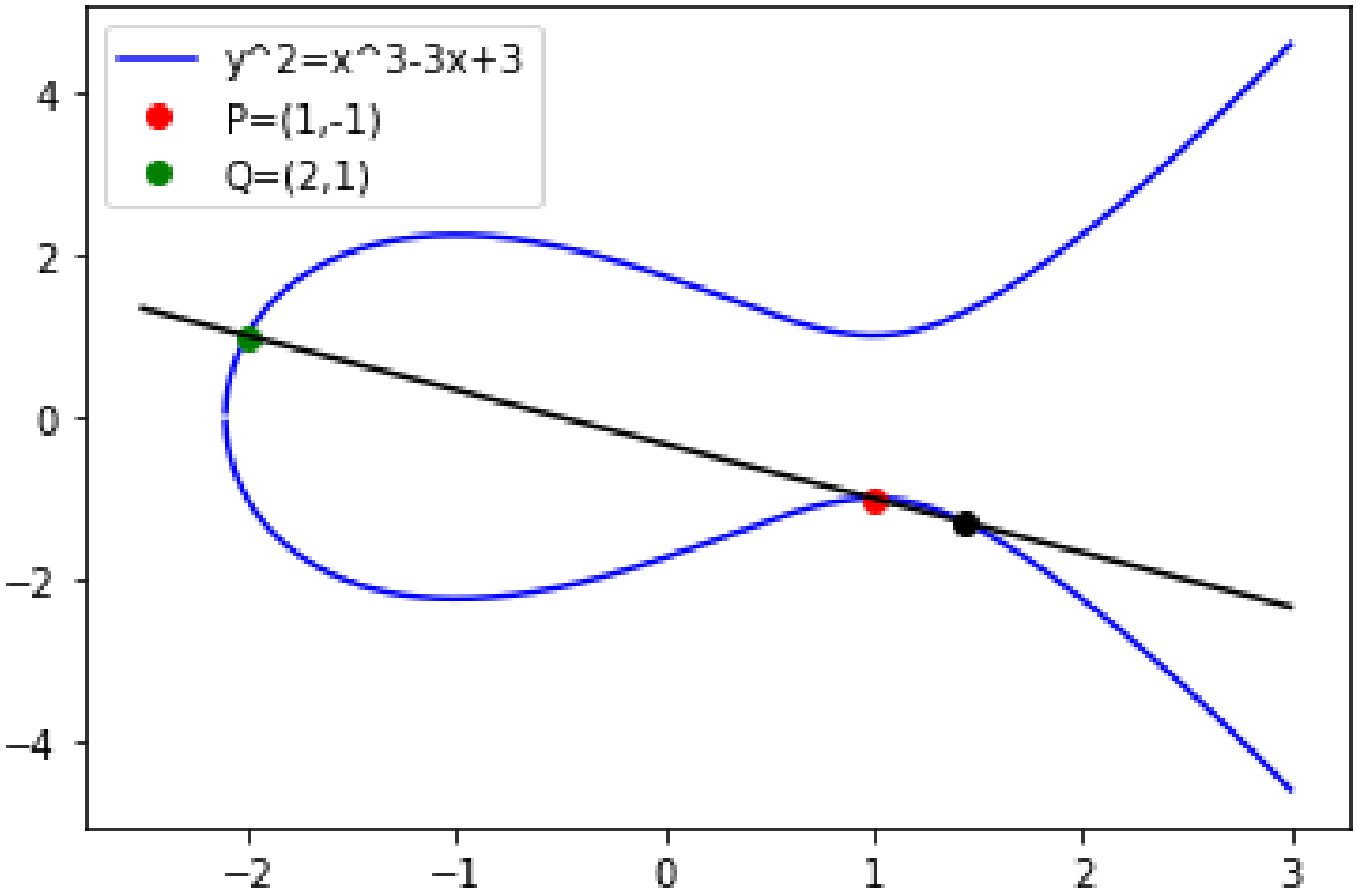

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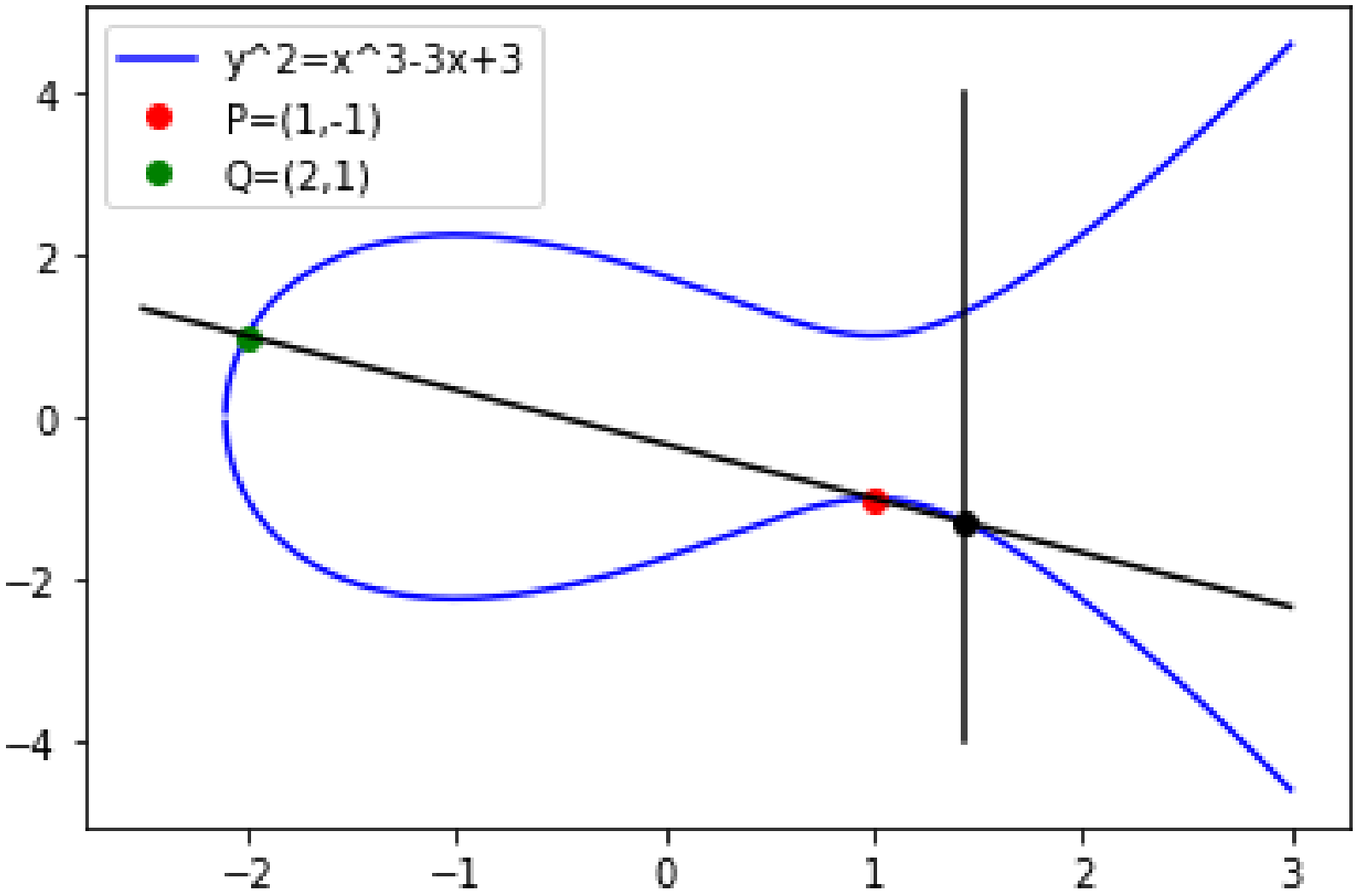

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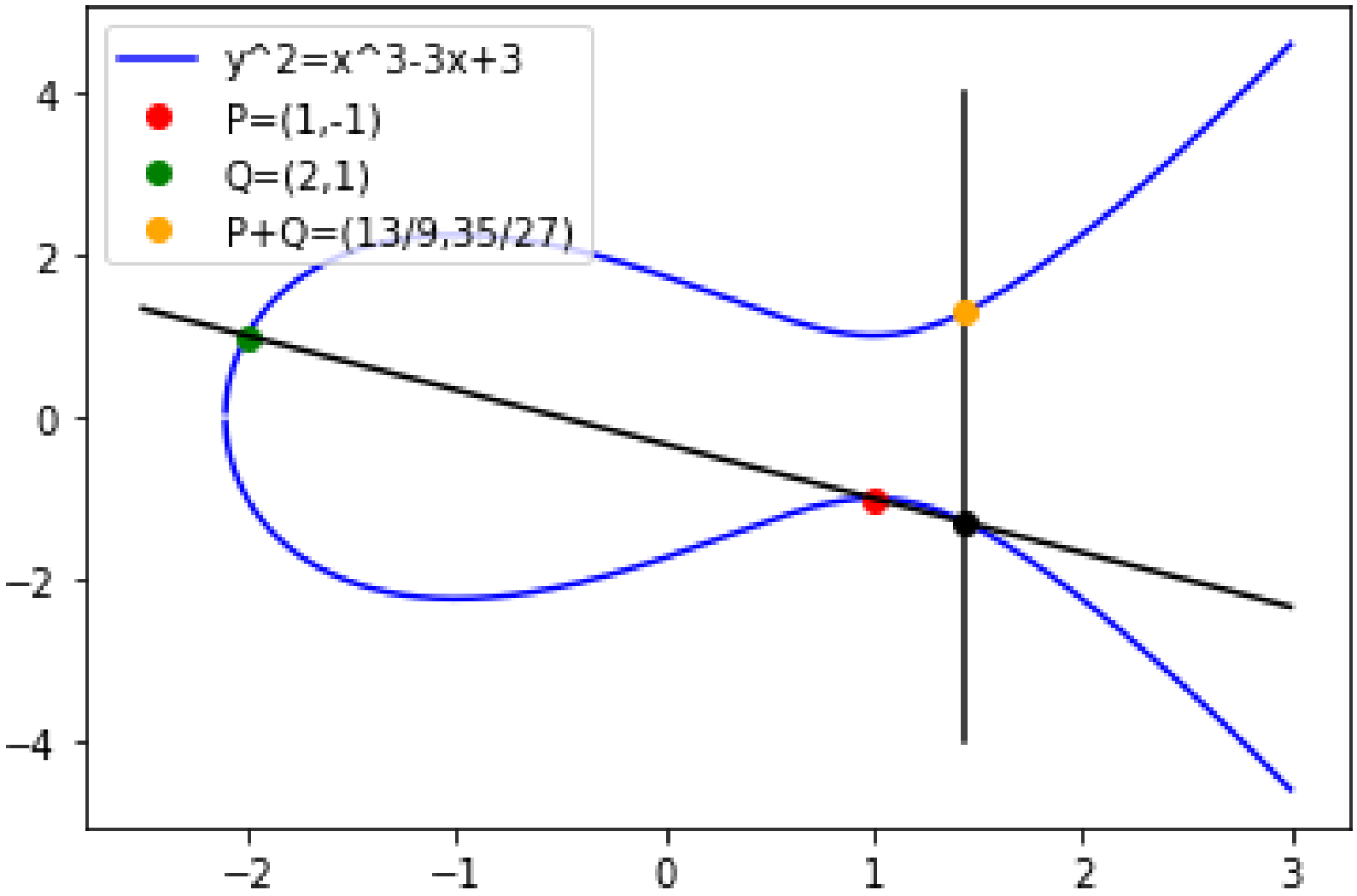

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The set of points $E(\mathbb{Q})$ is an abelian group.

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Structure

We have that $E(\mathbb{Q}) = E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$, where

- $E(\mathbb{Q})_{\text{tors}}$: torsion points $\rightarrow$ points of finite order such that

$$[n]P = \underbrace{P + \cdots + P}_{n \text{ times}} = O$$

(T is a finite group, easy to compute)

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- $\{P_1, \ldots, P_r\}$ set of points that generates the remaining of $E(\mathbb{Q})$.

Rank

r is called the rank of E. We know it is finite, but it is hard to compute.

Galois cohomology inerpretation

Fix a natural number m and let $\overline{\mathbb{Q}}$ be the algebraic closure of the rationals.

Torsion points

$E[m]$ are points $Q \in E(\overline{\mathbb{Q}})$ such that $[m]Q = O$. There are exactly m^2 of those points.

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We can do the same for every completion $\mathbb{Q}_\ell$:

Commutative diagram

$$\begin{array}{ccc} E(\mathbb{Q})/mE(\mathbb{Q}) & \xrightarrow{\kappa} & H^1(\mathbb{Q}, E[m]) \\ \downarrow \subset & & \downarrow \mathrm{loc}_\ell \\ E(\mathbb{Q}_\ell)/mE(\mathbb{Q}_\ell) & \xrightarrow{\kappa_\ell} & H^1(\mathbb{Q}_\ell, E[m]) \end{array}$$

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$$\mathrm{Sel}(\mathbb{Q}, E[m]) := \bigcap_{\ell \text{ prime}} \mathrm{loc}_\ell^{-1}\left(\kappa_\ell\left(E(\mathbb{Q}_\ell)/mE(\mathbb{Q}_\ell)\right)\right)$$

Absolute Selmer group

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We can make this construction considering all natural numbers m at the same time:

$$\mathrm{Sel}(\mathbb{Q}, E_{\mathrm{tors}}) = \bigcup_{m=1}^{\infty} \mathrm{Sel}(\mathbb{Q}, E[m]) \subset H^1(\mathbb{Q}, E_{\mathrm{tors}}(\overline{\mathbb{Q}}))$$

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Short exact sequence

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathrm{Sel}(\mathbb{Q}, E_{\mathrm{tors}}) \twoheadrightarrow Ш(E/\mathbb{Q}) \to 0$$

where $Ш(E/\mathbb{Q})$ is the Tate-Shafarevich group, conjecturally a finite group.

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where $Ш(E/\mathbb{Q})$ is the Tate-Shafarevich group, conjecturally a finite group. Then

$$E(\mathbb{Q}) \otimes \mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q}/\mathbb{Z})^r \Rightarrow \mathrm{Sel}(\mathbb{Q}, E_{\mathrm{tors}}) \cong (\mathbb{Q}/\mathbb{Z})^r \times Ш(E/\mathbb{Q})$$

Absolute p-Selmer group

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Fix a prime number p

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Canonical split

The absolute Selmer group can be computed as

$$\mathrm{Sel}(\mathbb{Q}, E_{\mathrm{tors}}) = \prod_{p \text{ prime}} \mathrm{Sel}(\mathbb{Q}, E[p^\infty])$$

Galois representation

Generalisation

The construction of Selmer groups is not exclusive from elliptic curves, but can be obtained from every Galois representation.

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A Selmer group is the subgroup of the global Galois cohomology group cut out by some local conditions.

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Key idea

A Selmer group is the subgroup of the global Galois cohomology group cut out by some local conditions.

Class groups

The class group of a number field can be computed from the Selmer group of $\mathbb{Q}/\mathbb{Z}$ with trivial Galois action. By class field theory,

$$\mathrm{Sel}(K, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Cl}(K)$$

L-functions

Definition

They can be defined for every (p-adic) Galois representation. They can be constructed as an infinite product

$$L: \ \mathbb{C} \to \mathbb{C}, \ s \mapsto \prod_{\ell \text{ prime}} \mathrm{Eul}_\ell(s)$$

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Properties

- This product converges absolutely when $\Re(s) \gg 0$.
- In some cases of interest, it can be meromorphically extended to all $\mathbb{C}$.

Example of L-functions

Riemann zeta function

It is obtained from the roots of unity, working over $\mathbb{Q}$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\ell \text{ prime}} \frac{1}{1-\ell^{-s}}, \quad \Re(s) > 1$$

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Dedekind zeta functions

We can generalise this construction to number fields

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{\left(\#(\mathcal{O}_K/I)\right)^{-s}} = \sum_{\mathfrak{p} \text{ prime}} \frac{1}{1-\left(\#(\mathcal{O}_K/p)\right)^{-s}}, \quad \Re(s) > 1$$

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L-function of an elliptic curve

$$L(E,s) = \prod_{\ell \text{ prime}} \frac{1}{1-a_\ell \ell^{-s} + \ell^{1-2s}}, \quad \Re(s) > 3/2$$

where a_ℓ is defined as:

$$a_\ell = \ell + 1 - \#E(\mathbb{F}_\ell)$$

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Analytic class number formula (Dirichlet, Dedekind)

$\zeta_K(s)$ has a pole at $s = 1$ and the residue

$$\mathrm{Res}_{s=1}\zeta_K = \left(\frac{2^{r_1}2\pi^{r_2}R_K}{\omega_K\sqrt{|\Delta_K|}}\right) h_k$$

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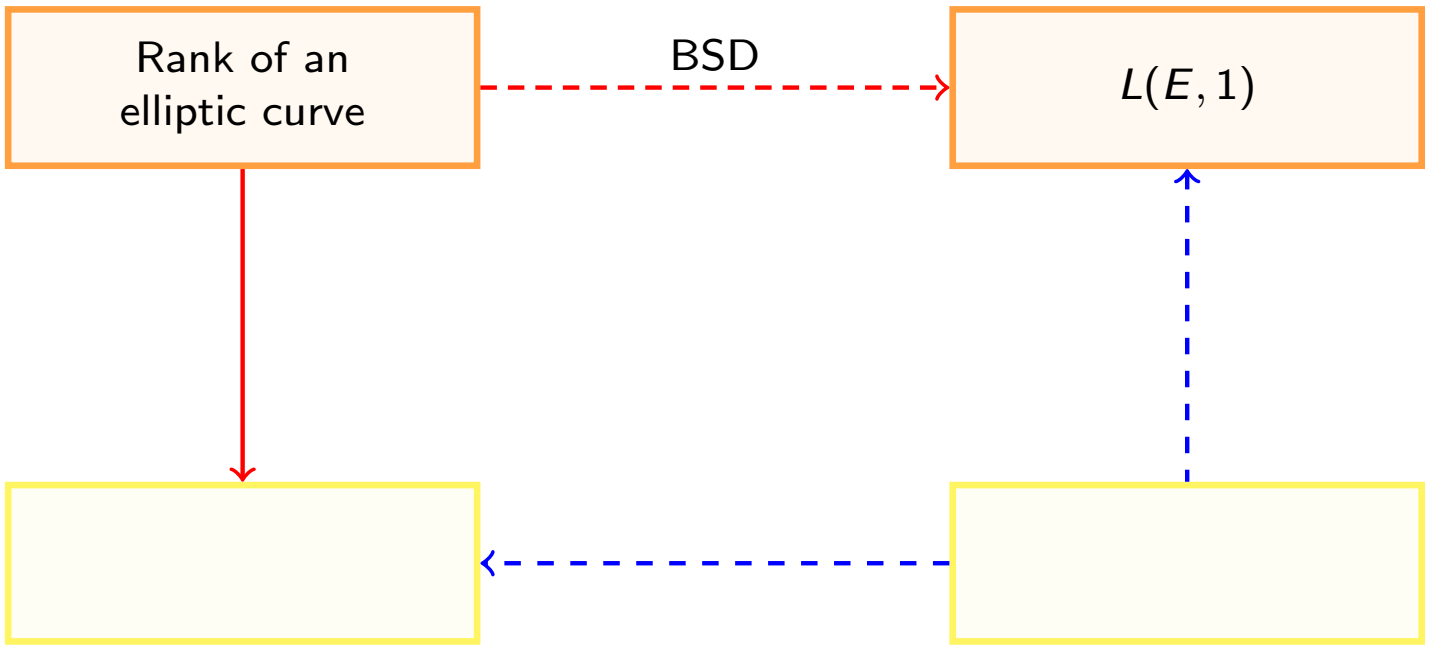
Birch and Swinnerton-Dyer conjecture (BSD)

- $\mathrm{ord}_{s=1}L(E,s) = \mathrm{rank}(E(\mathbb{Q}))$
- The first non-zero term of the Taylor expansion of $L(E,s)$ at $s = 1$ has an arithmetic interpretation.

$$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega_E \cdot R_E \cdot \#Ш(E/\mathbb{Q}) \cdot \mathrm{Tam}(E)}{(\#E(\mathbb{Q})_{\mathrm{tors}})^2}$$

Strategy for BSD



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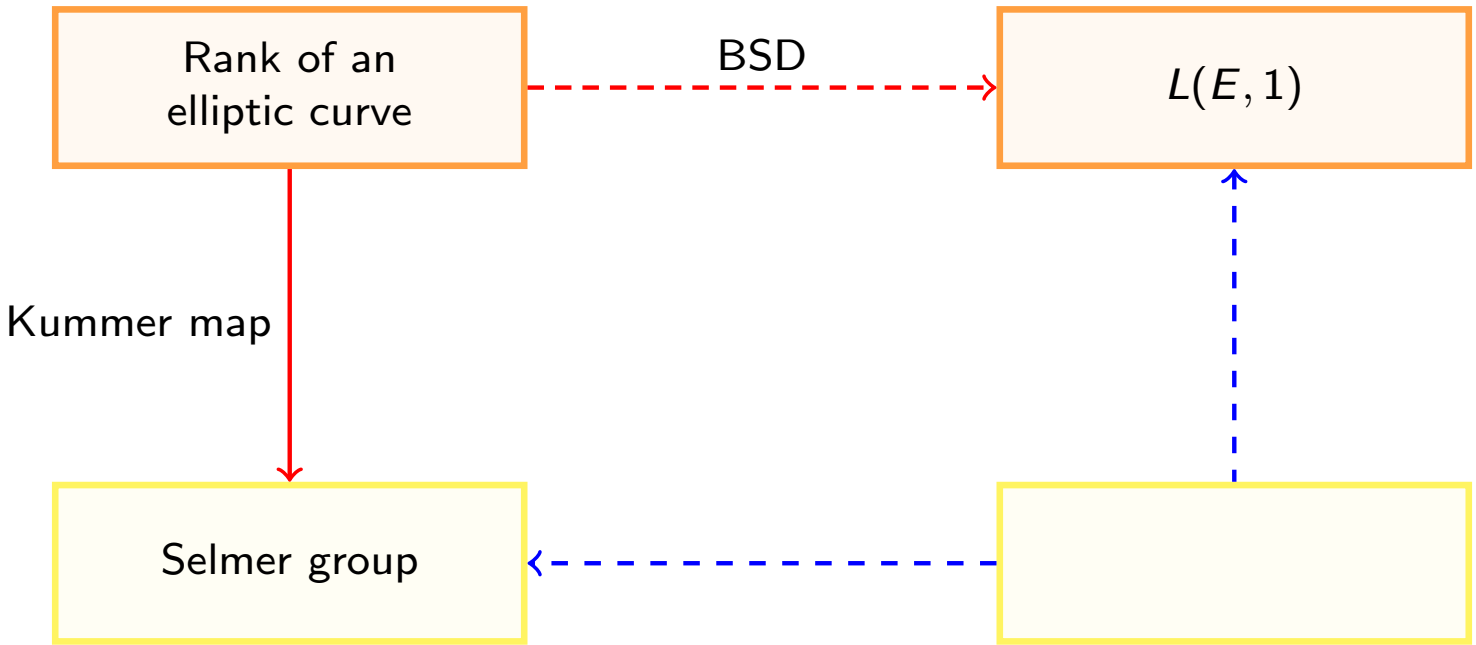

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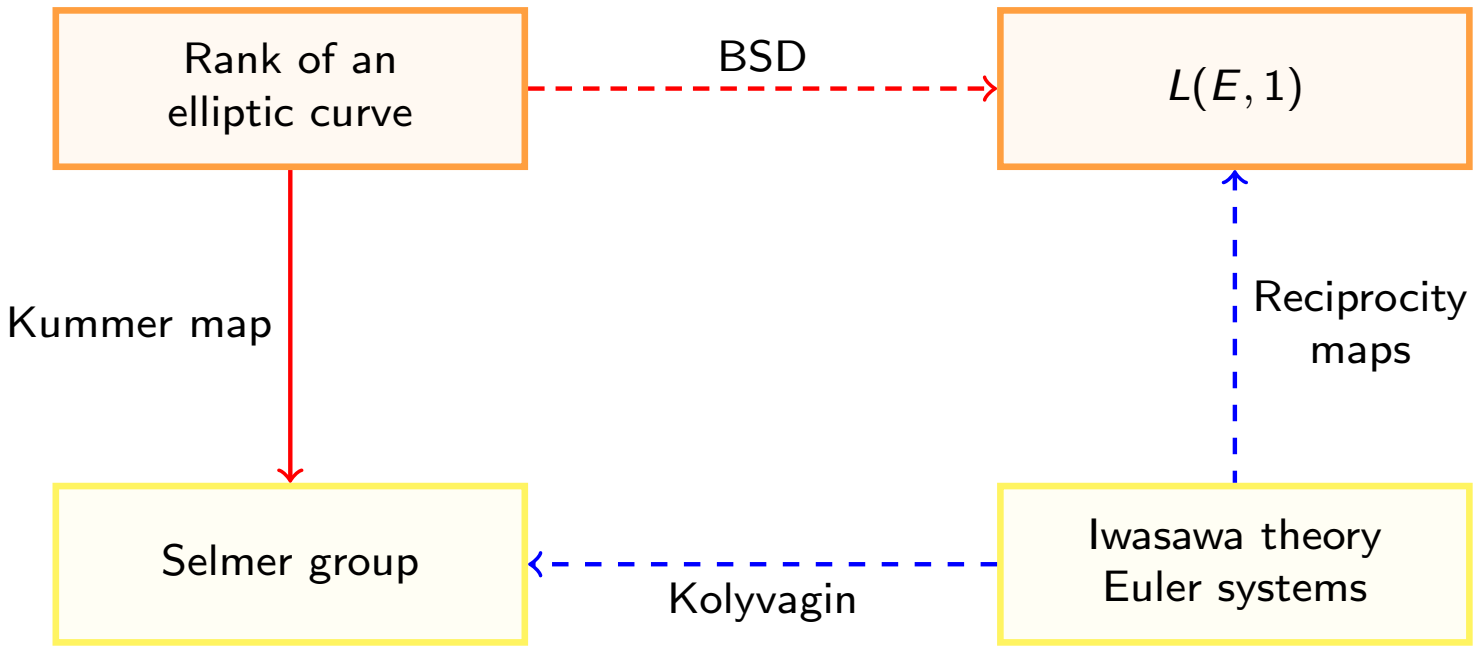

Iwasawa theory

Idea of Iwasawa

Instead of studying $E(\mathbb{Q})$, it is easier to study $E(\mathbb{Q}_\infty)$, where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension.

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- Consider the field $\mathbb{Q}(\zeta_{p^{n+1}})$, where $\zeta_{p^{n+1}} = e^{\frac{2\pi i}{p^{n+1}}}$.

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Structure of the rational points

- $E(\mathbb{Q}_\infty)$ is a module over the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[T]]$.
- Modules over this ring behave nicely: there is a structure theorem describing them (up to finite index).

Euler systems

Euler systems

Let T be a p-adic representation. An Euler system is a collection

$$\mathbf{c} = \{c_n \in H^1(\mathbb{Q}(\zeta_n), T) : \ n \in \mathbb{N}\}$$

satisfying norm-compatibilty relations, defined using Euler factors.

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Construction of Euler systems

- Only a handful of Euler systems are known.
- The construction of new ones is a very active research area.
- They usually come from the relation between Galois representations and modular forms/automorphic forms.

Control of the Selmer group

Kolyvagin derivative

It is a process that, from an Euler system, constructs new classes which bound the Selmer group.

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Theorem (Kolyvagin, Kato)

Under mild assumptions, if Kato's Euler system satisfies that $\mathrm{loc}_p(z_1) \neq 0$. Then the Selmer group

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BSD in rank 0

BSD holds when the analytic rank is 0.

$$L(E,1) \neq 0 \Rightarrow \mathrm{loc}_p(z_1) \neq 0 \Rightarrow \#\mathrm{Sel}(\mathbb{Q}, E[p^\infty]) < \infty \Rightarrow \mathrm{rank}(E(\mathbb{Q})) = 0$$

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BSD in rank 1

The argument can be generalised when the analytic rank is 1 by using Heegner points.

Structure of the Selmer group

Theorem (C.-H. Kim 2025)

From Kato's Euler system, one can define a sequence of ideas $\Theta_i \subset \mathbb{Z}_p$. For every $i \in \mathbb{Z}_{\geq 0}$ of certain parity (depending on E)

$$\Theta_i = \mathrm{Fitt}^i\left(\mathrm{Sel}(\mathbb{Q}, E[p^\infty])^\vee\right)$$

where $\mathrm{Fitt}^i\left(*\right)$ is the i^{th} Fitting ideal.

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- If T is not self-dual, the equality holds for all $i \in \mathbb{Z}_{\geq 0}$.

These conditions are enough to determine the Selmer group up to isomorphism.

It generalises Kim's result to compute the Selmer group of an elliptic curve over an abelian extension.

Kurihara numbers

Modularity theorem (Wiles, et.al., 1995) and modular symbols

For every elliptic curve (defined over $\mathbb{Q}$), there is a modular form f associated to E, which is an holomorphic function

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Kurihara numbers can be effectively computed and Θ_i is obtained as

$$\Theta_i := \sum_{n \in \mathcal{N}_i} \delta_n \subset \mathbb{Z}_p$$

Structure of Iwasawa modules

Iwasawa Selmer group

Recall that Iwasawa theory studies the points in $E(\mathbb{Q}_\infty)$. That can be done using an Iwasawa Selmer group $\mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])$.

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Theorem (A. 2026)

Assume some mild big image assumptions. From Kato's Euler system, one can define ideals $\boldsymbol{\Theta}_i$ for every $i \in \mathbb{Z}_{\geq 0}$ satisfying that

$$\boldsymbol{\Theta}_i = \mathrm{Fitt}^i\left(\mathrm{Sel}_p(\mathbb{Q}_\infty, E[p^\infty])^\vee\right)$$

That gives a description of the Selmer group up to finite index.

Idea of the proof

- The classical theory of Euler systems works with finite Selmer groups

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- Following work of Sweeting and Loeffler-Zerbes, we can consider hyperprimes, coming from non-standard analysis, to define the local conditions.

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- Following work of Sweeting and Loeffler-Zerbes, we can consider hyperprimes, coming from non-standard analysis, to define the local conditions.

- They are sequences of primes $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}}$ identified with an equivalence relation defined in terms of an ultrafilter.

Idea of the proof

- The classical theory of Euler systems works with finite Selmer groups

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- They allow us to work direcly with Iwasawa Selmer modules, where we can take advantage of the structure theorem of finitely generated modules.

Thank you!