Arithmetic of the twisted L-values of elliptic curves

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Introduction

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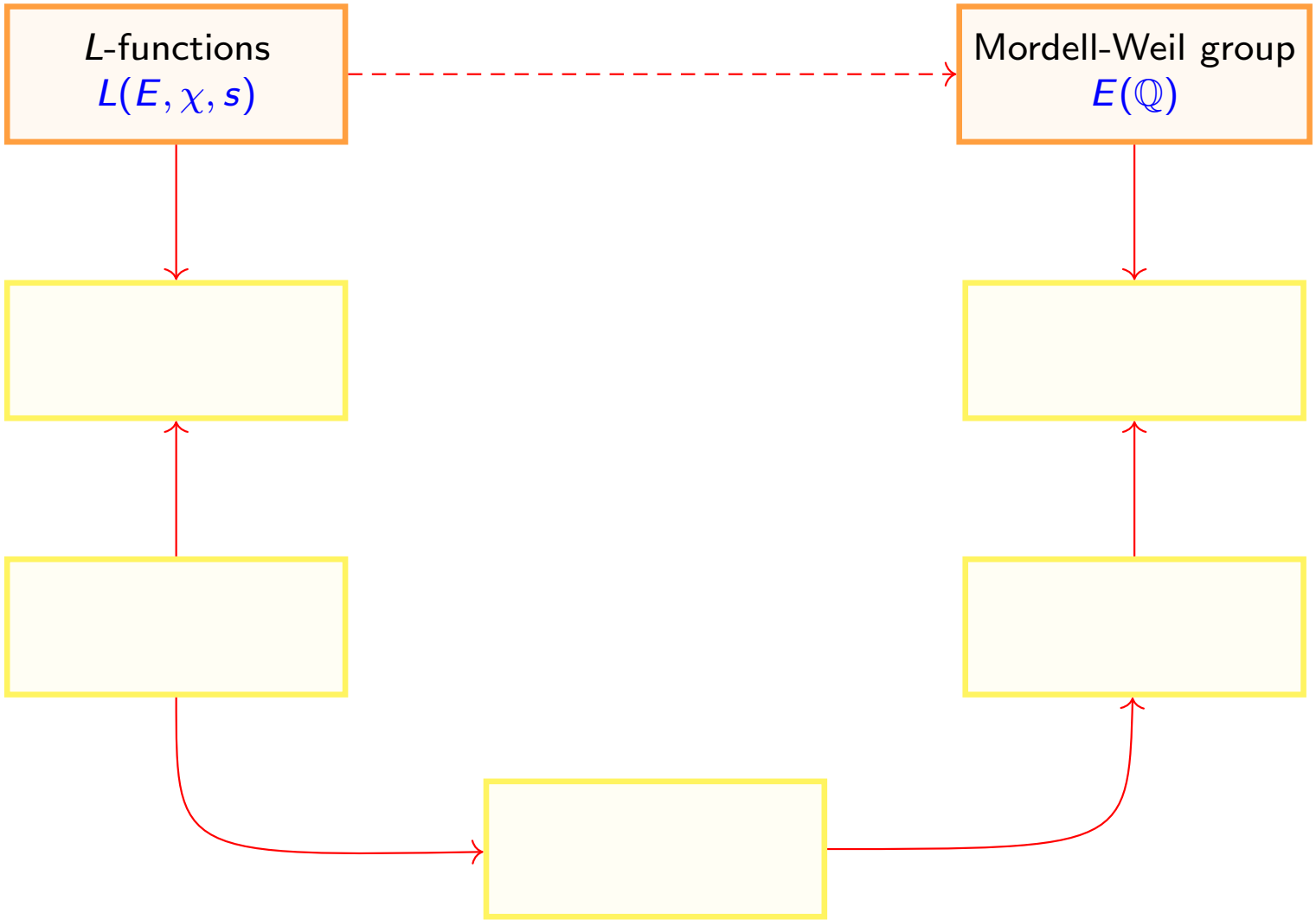
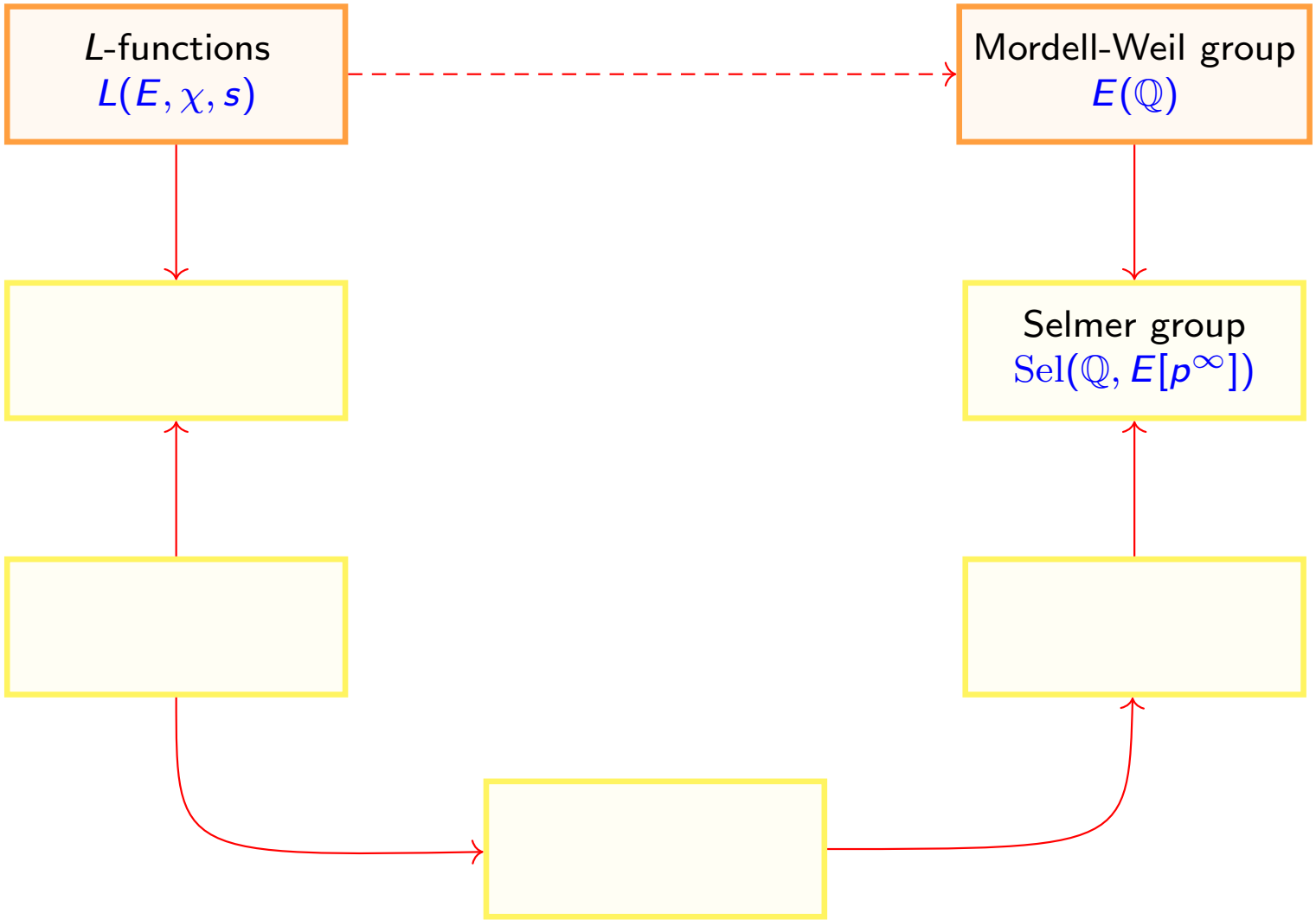
Generalisation to abelian extensions

The above mentioned set also determines the rank of $E(K)$ and the Galois structure of $Ш(E/K)$ for most abelian extensions $K/\mathbb{Q}$.

General picture



Selmer group



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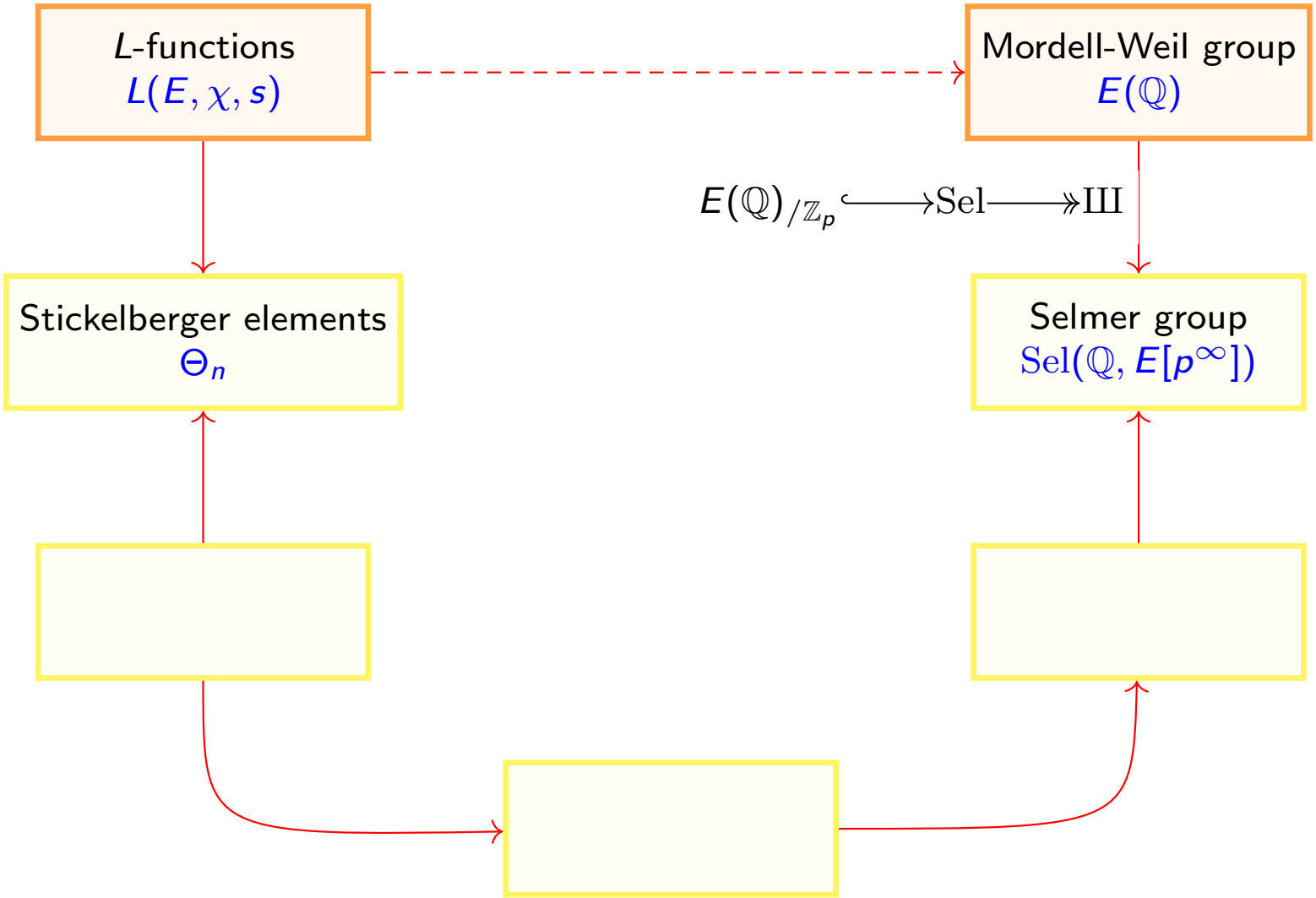
- In this talk, we will assume that $Ш(E)$ is finite, so $r_{\mathrm{alg}} = r_{\mathrm{sel}}$.

Stickelberger elements



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$$\Theta_n := \sum_{\chi \bmod n} \frac{L(E,\chi,1)}{\tau(\chi)\Omega^{\pm}} e_\chi \in \overline{\mathbb{Q}}_p[\mathcal{G}_n]$$

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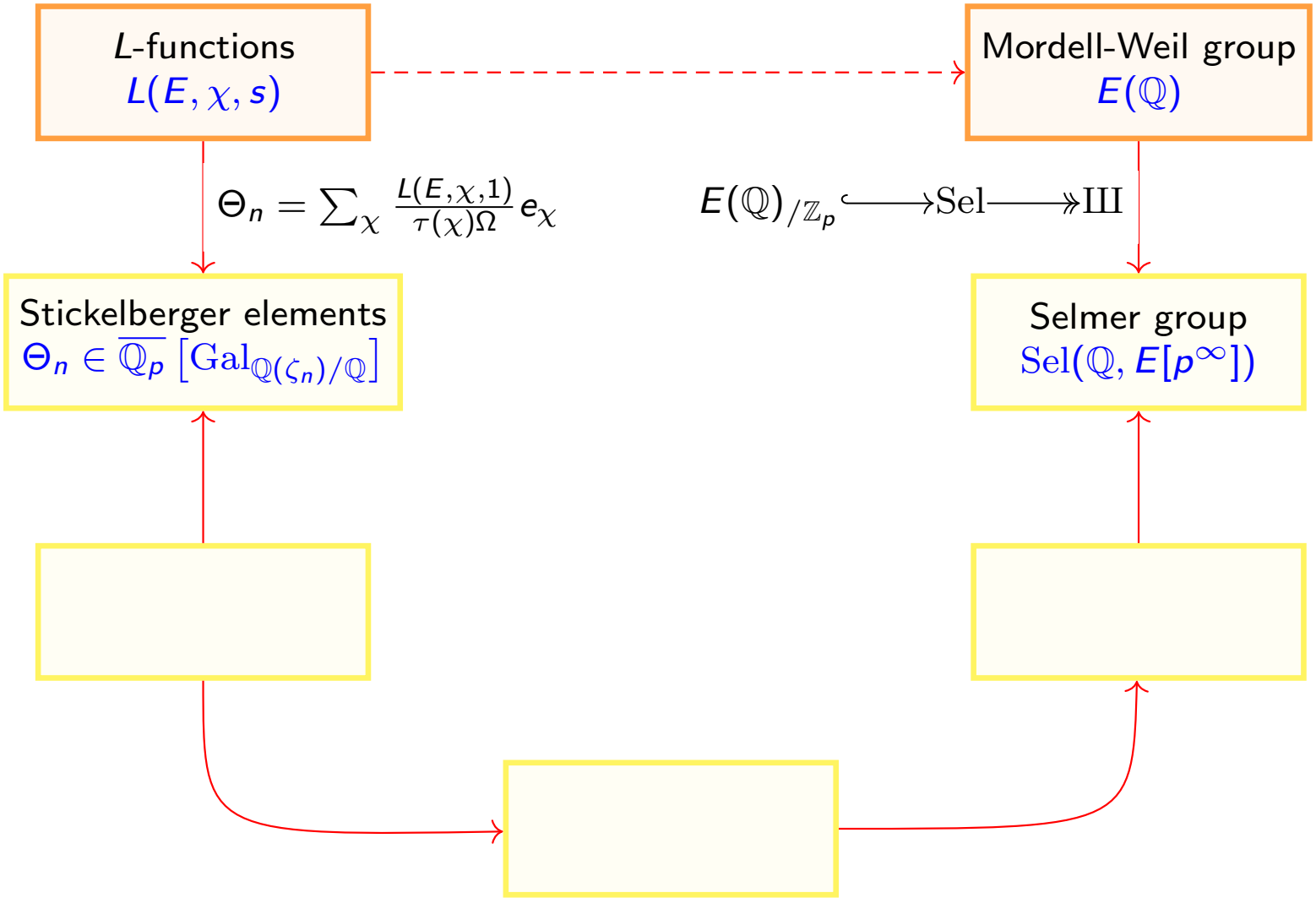
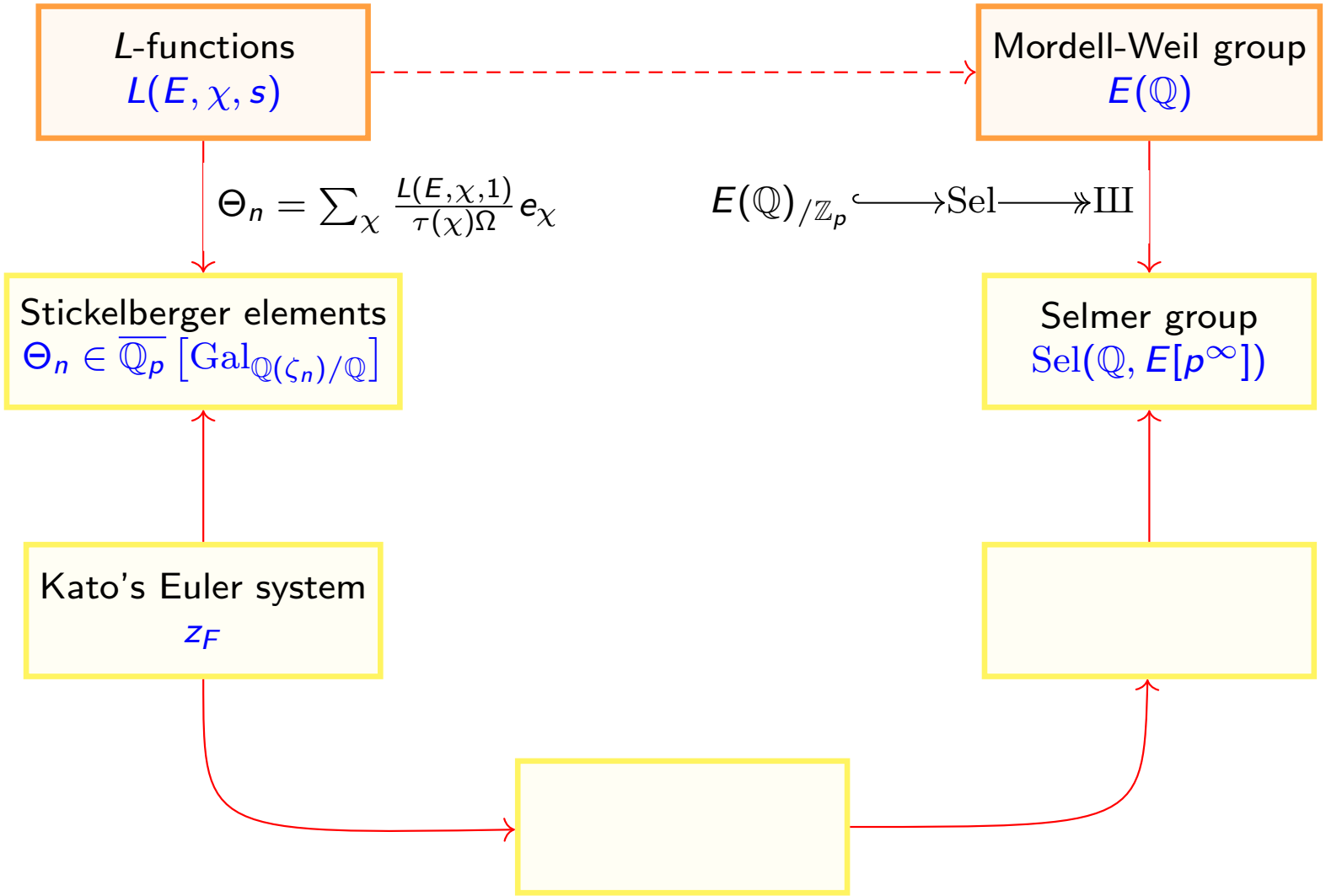


Kato's Euler system



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$$\exp^* : H^1(F, T_pE) \to \mathbb{Q}(\zeta_n), \quad \boxed{z_{\mathbb{Q}(\zeta_n)} \mapsto (*)\Theta_n(\zeta_n)}$$

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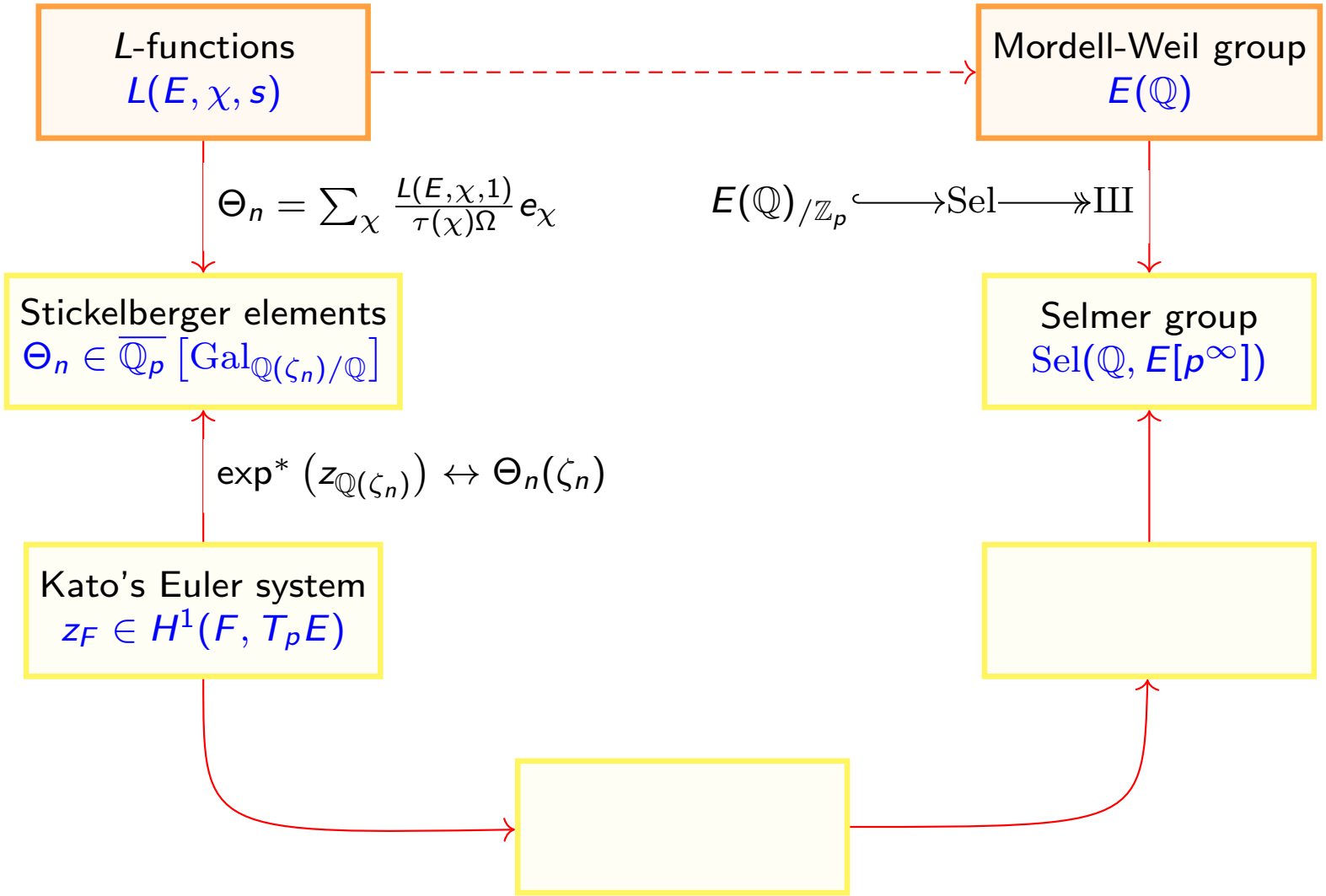
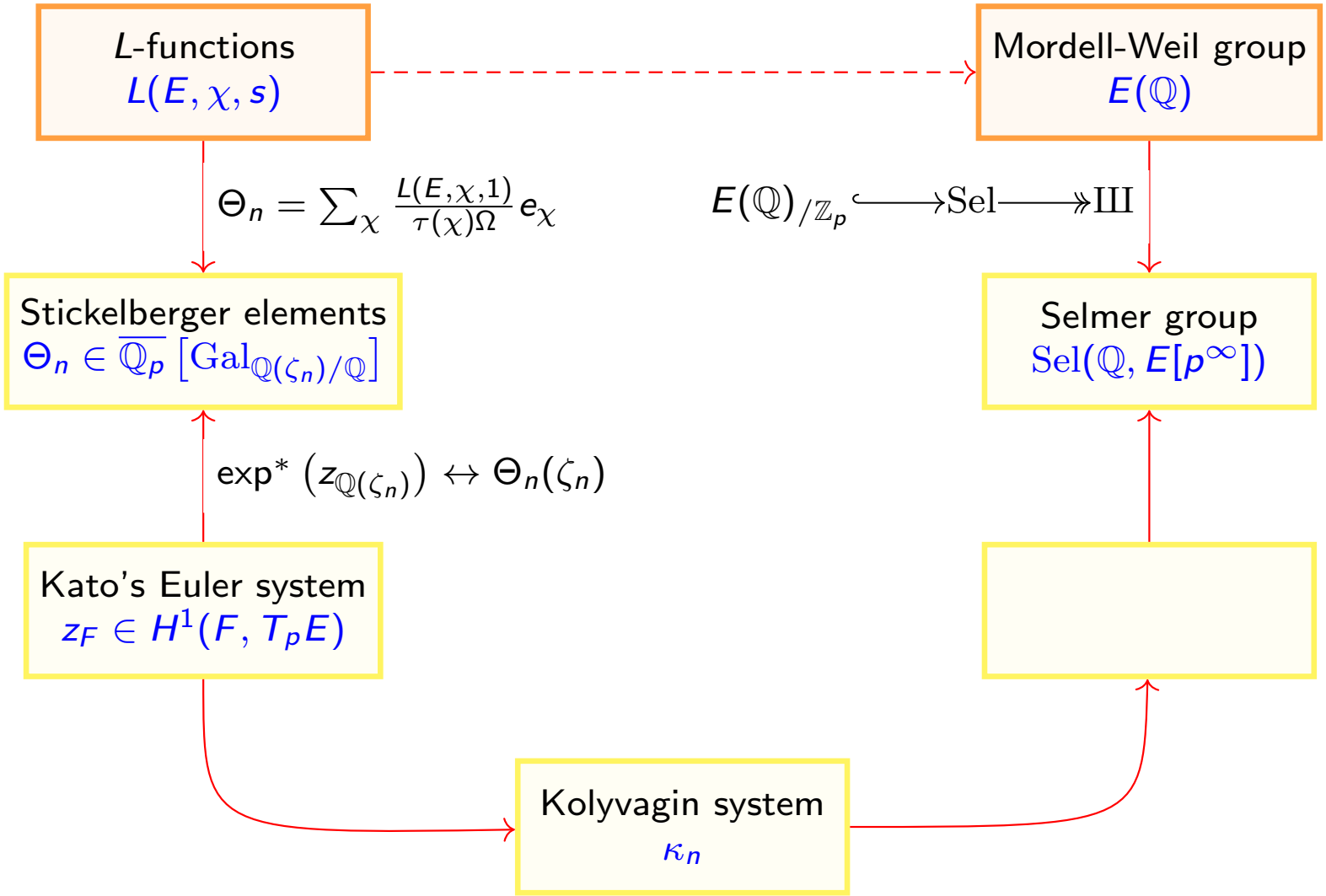


Kolyvagin systems



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- These conditions are very rigid: there is only one Kolyvagin system up to constant.

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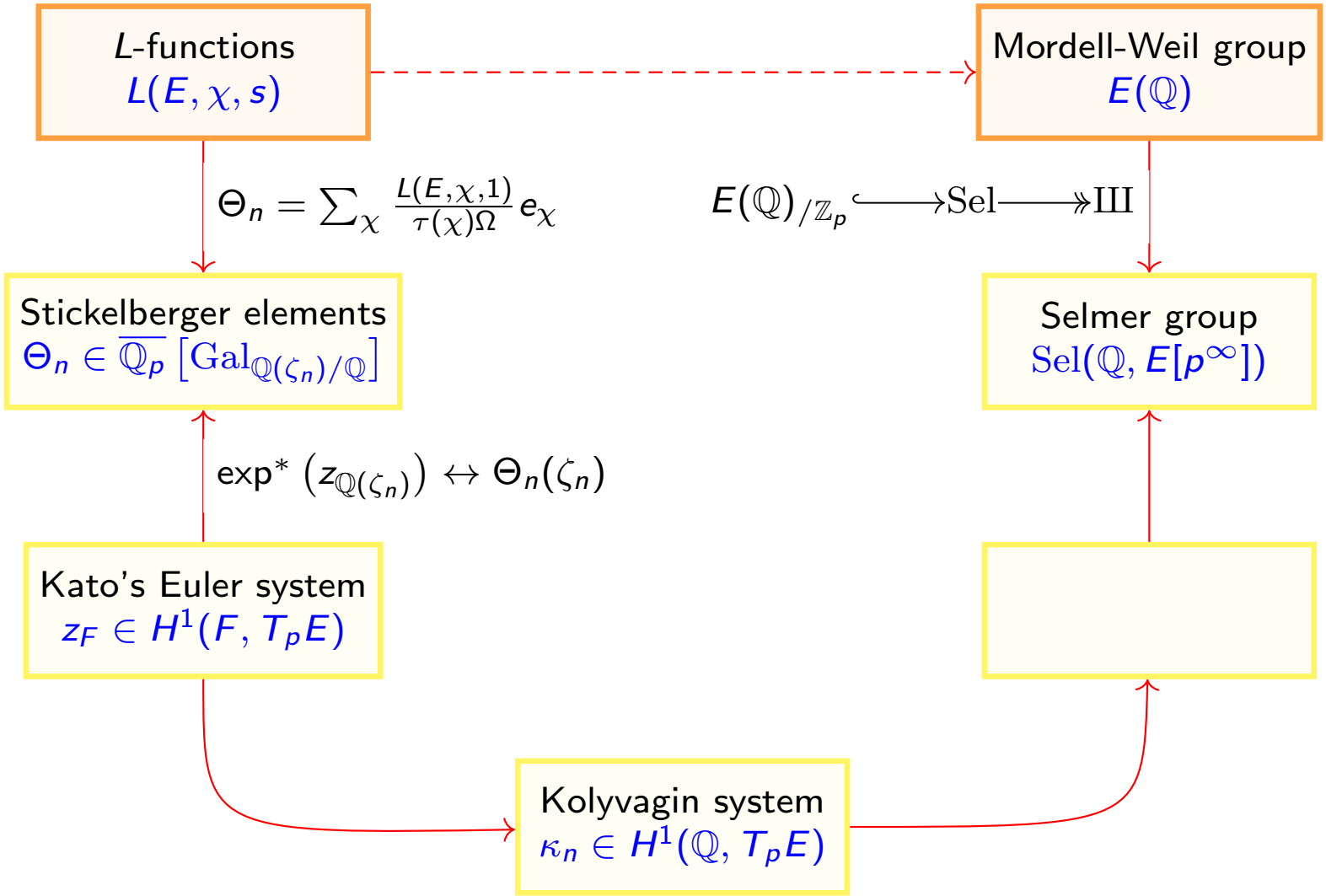

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From Euler systems to Kolyvagin systems

The Kolyvagin derivative defines a map

$$\Phi : \{\text{Euler systems}\} \to \{\text{Kolyvagin systems}\}$$

satisfying that, if $\Phi(z) = \kappa$, then $\kappa_1 = z_{\mathbb{Q}}$.

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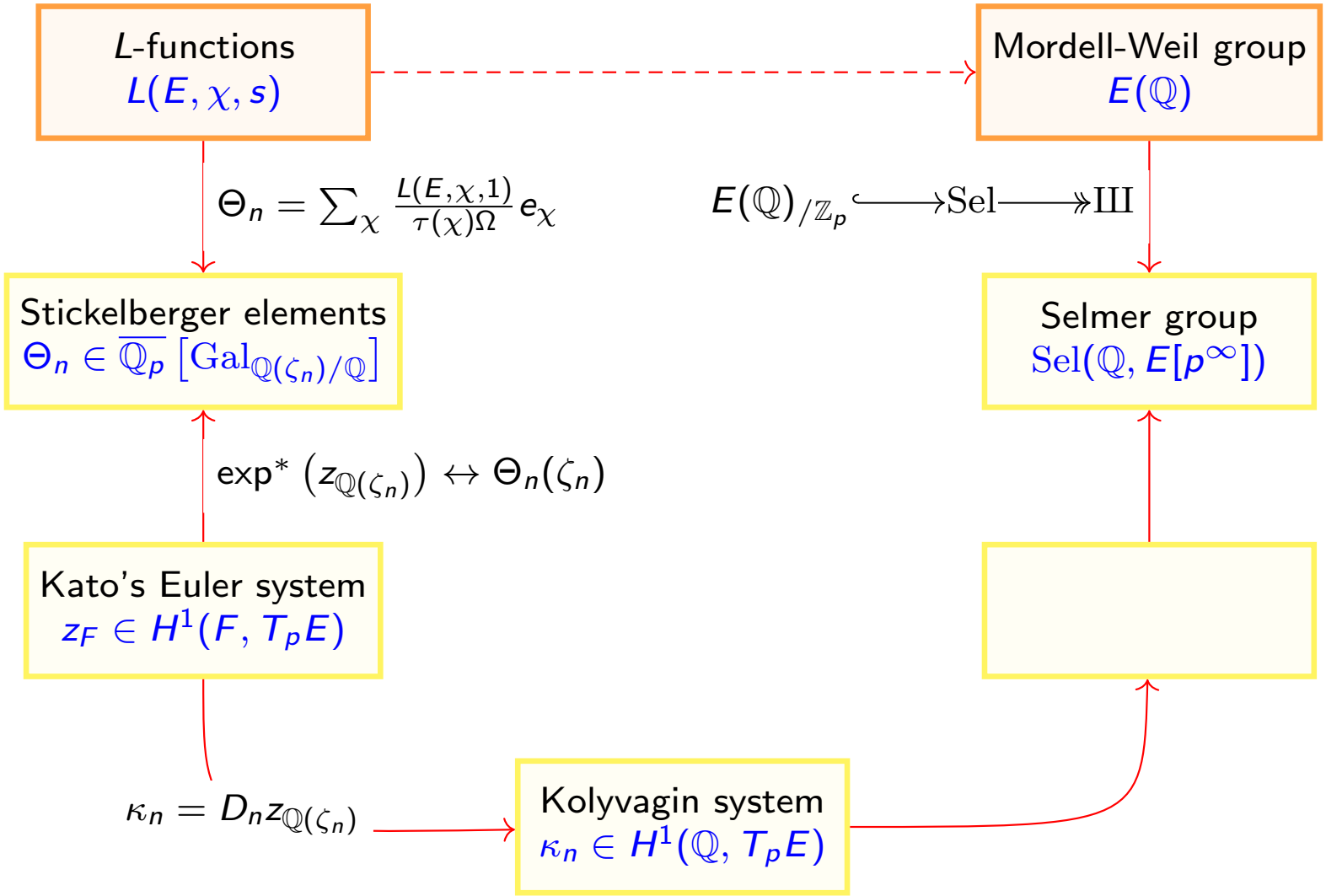
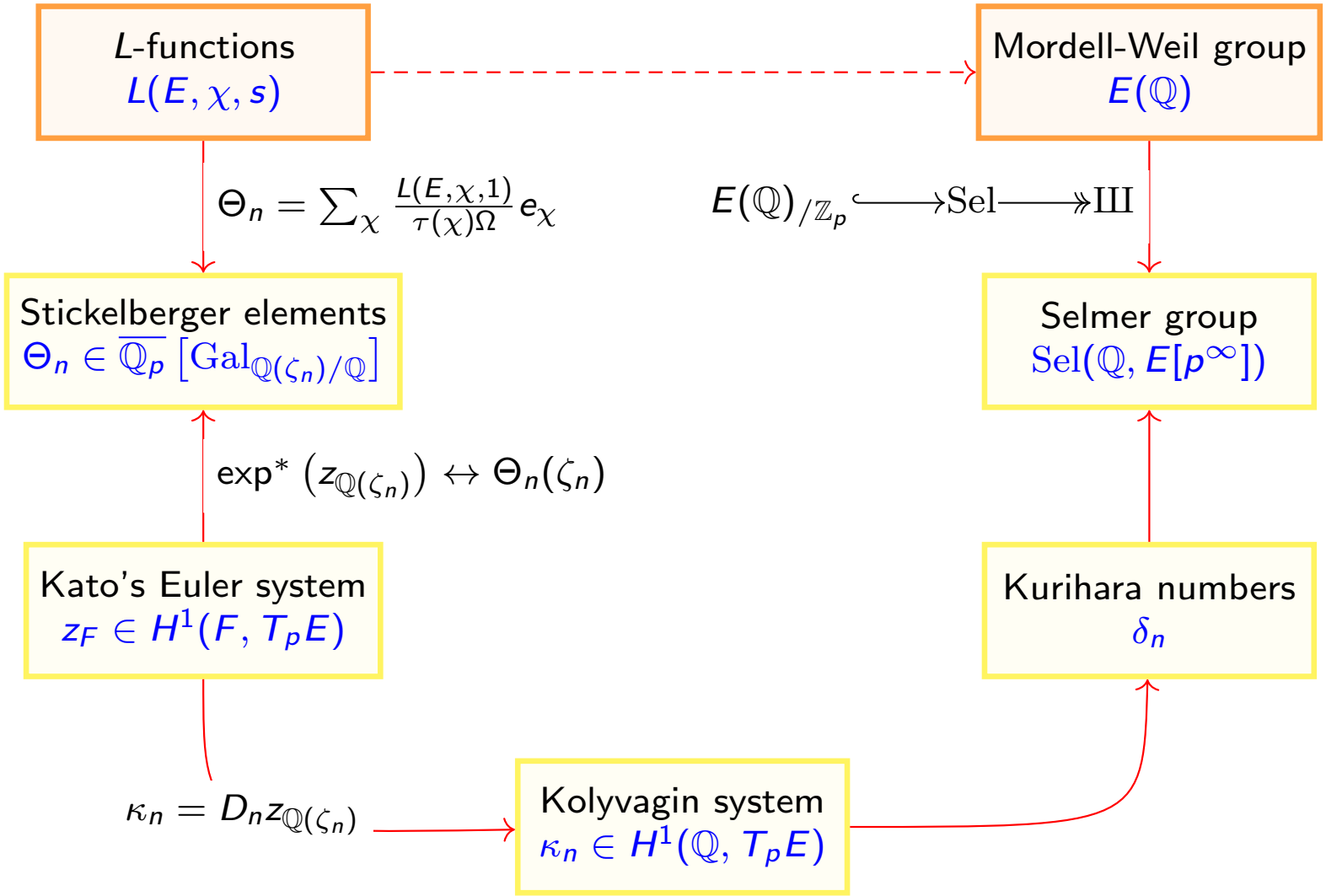


Kurihara numbers



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- They depend on η_ℓ, but their p-adic valuation is independent of these choices.

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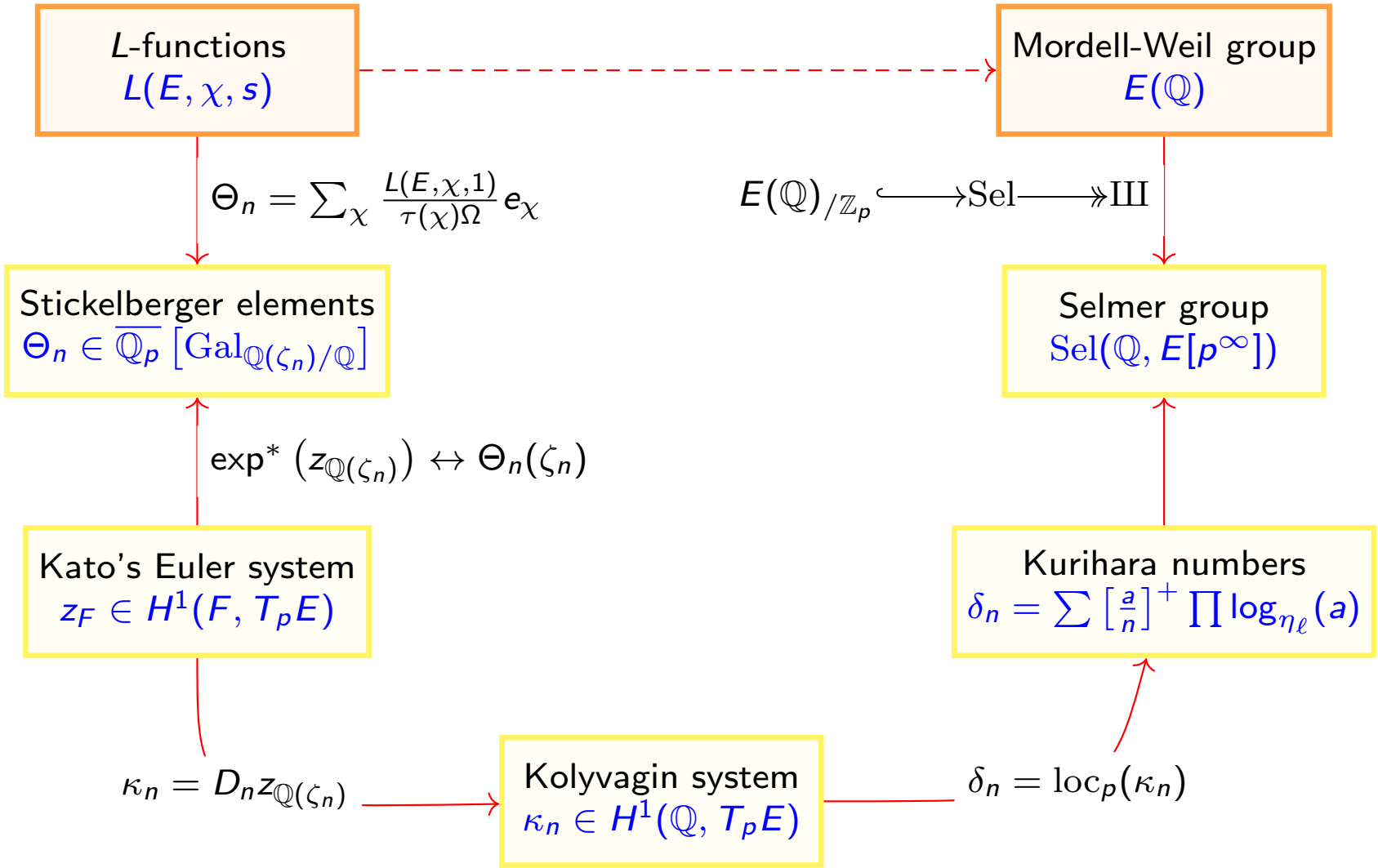
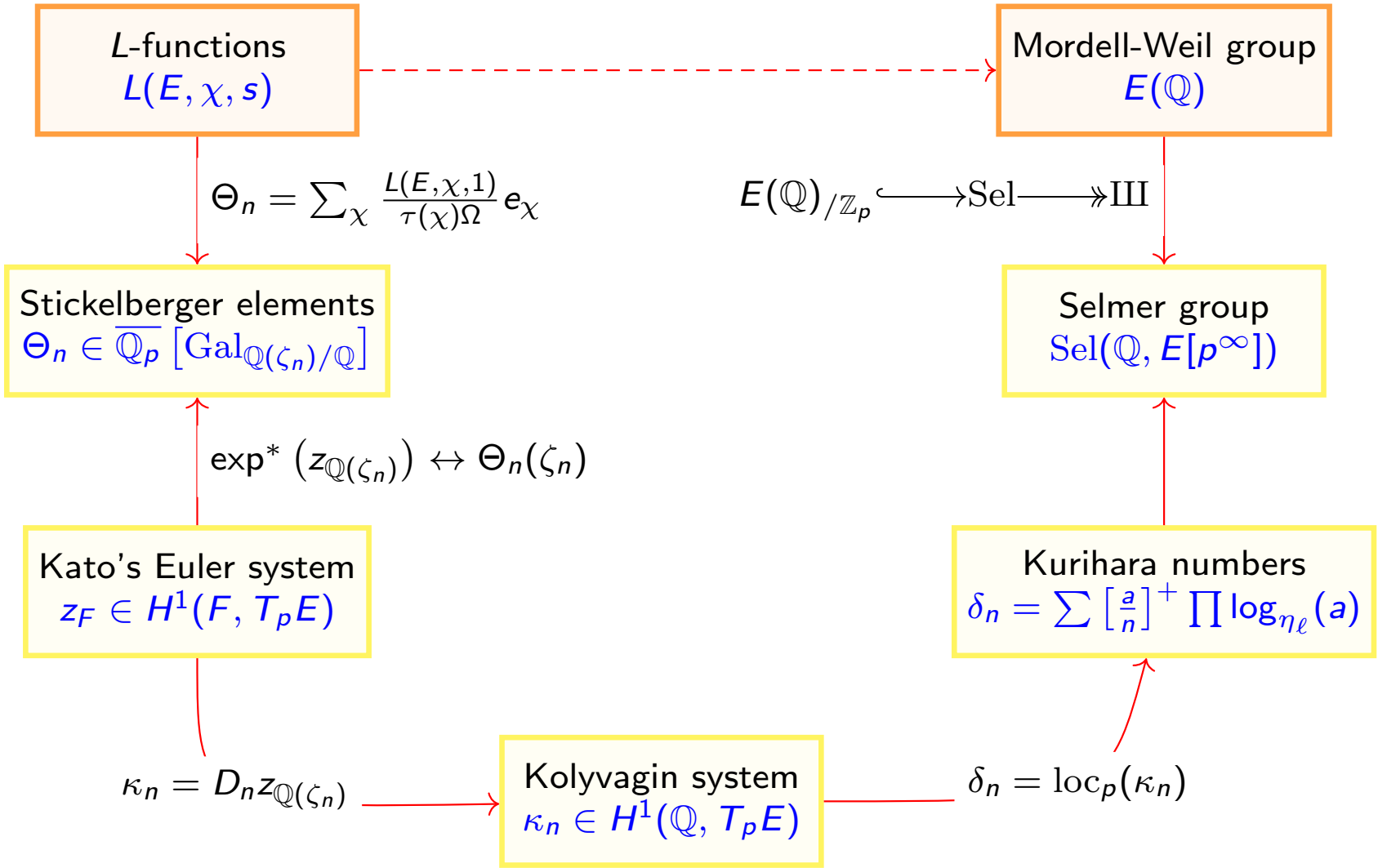


Structure of the Selmer group



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Fitting ideals

Let M be a finitely generated $\mathbb{Z}_p$-module. The structre theorem gives an isomorphism

$$M \cong \mathbb{Z}/p^{\alpha_1} \times \cdots \times \mathbb{Z}/p^{\alpha_r}$$

where $\alpha_i \in \mathbb{N} \cup \infty$. Assume $\alpha_1 \leq \cdots \leq \alpha_r$.

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$$\theta_i = \langle \{\delta_n : \ \nu(n) = i\} \rangle \subset \mathbb{Z}_p$$

- These ideals are related to the Fitting ideals of the Selmer group.

Fitting ideals

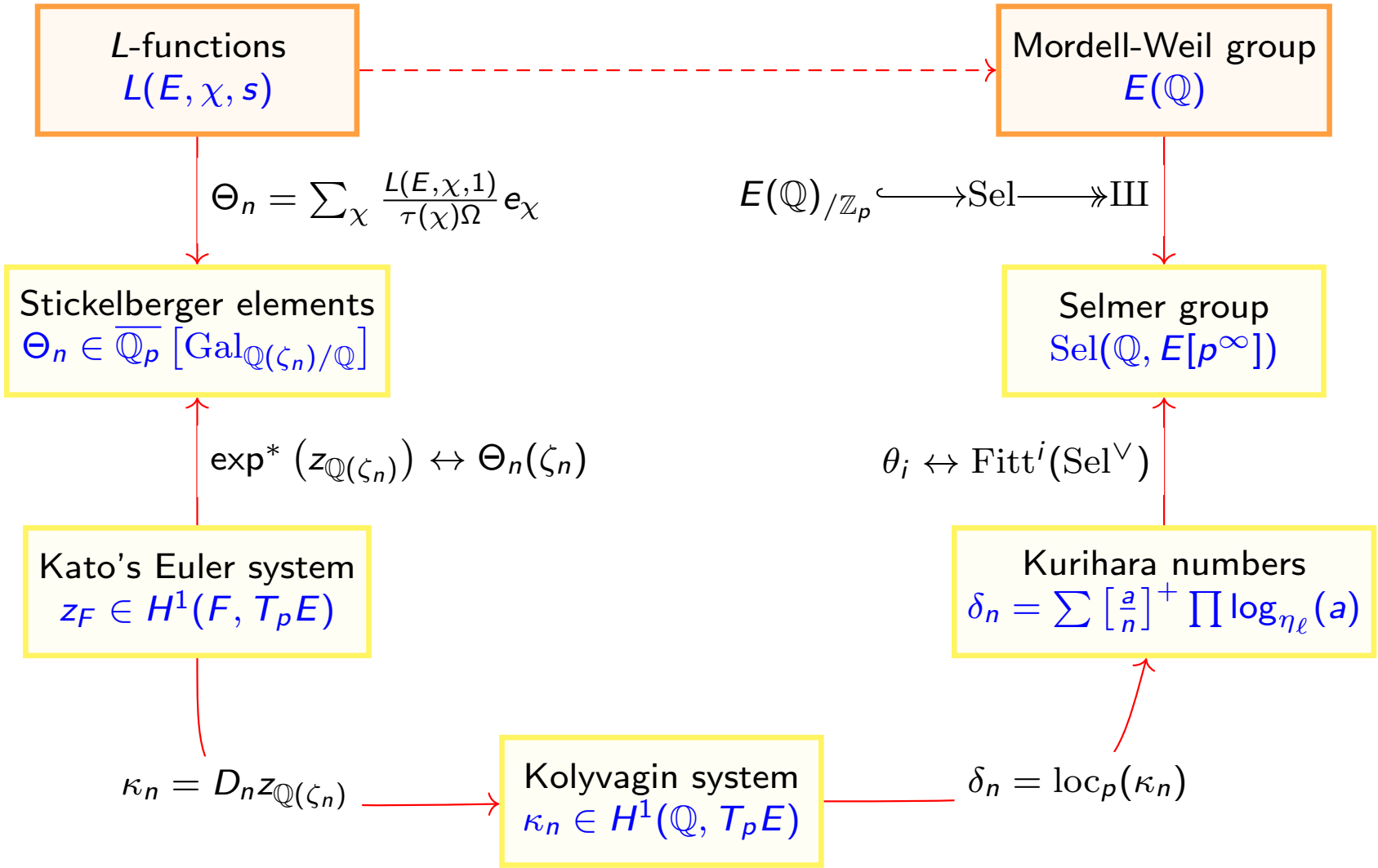
Let M be a finitely generated $\mathbb{Z}_p$-module. The structre theorem gives an isomorphism

$$M \cong \mathbb{Z}/p^{\alpha_1} \times \cdots \times \mathbb{Z}/p^{\alpha_r}$$

where $\alpha_i \in \mathbb{N} \cup \infty$. Assume $\alpha_1 \leq \cdots \leq \alpha_r$. Then

$$\begin{cases} \mathrm{Fitt}^i(M) = \prod_{k=1}^{r-i} p^{\alpha_k} & \text{if } 0 \leq i < r \\ \mathrm{Fitt}^i(M) = \mathbb{Z}_p & \text{if } i \geq r \end{cases}$$

Structure of the Selmer group



Main result

Theorem (C-H. Kim, 2025)

Let $p \geq 5$ satisfying that

- $G_{\mathbb{Q}}$ acts surjectively on $T_p E$.
- p divides neither the Tamagawa numbers nor the Manin constant
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Then,

$$\begin{cases} \theta_i = \mathrm{Fitt}^i(\mathrm{Sel}(\mathbb{Q}, E[p^\infty])^\vee) & \text{if } (-1)^i = \omega(E) \\ \theta_i = 0 & \text{if } (-1)^i \neq \omega(E) \end{cases}$$

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The above theorem, together with the Cassels-Tate pairing, can determine the full structure of the Selmer group.

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The values δ_n for other square-free n will determine the group structure of $Ш(E)[p^\infty]$.

Arithmetic over an abelian extension $K/\mathbb{Q}$

Assumptions on $K/\mathbb{Q}$

- The degree $[K:\mathbb{Q}]$ is prime to p.
- $K/\mathbb{Q}$ is unramified at p and at every bad prime of E.
- We call c the conductor of $K/\mathbb{Q}$.

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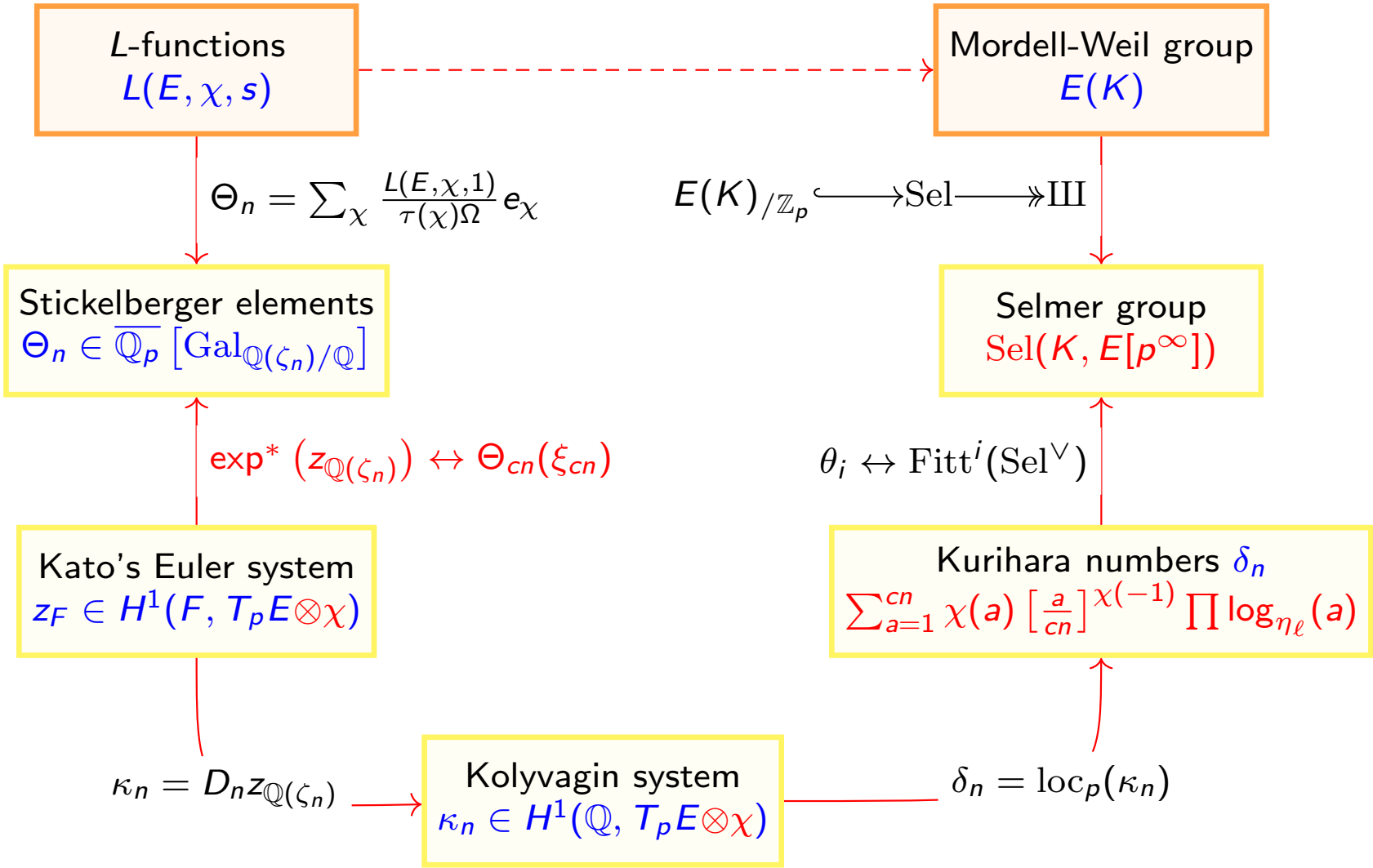

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$$\mathrm{Sel}(K, E[p^\infty]) =' \bigoplus_{\chi} Sel(\mathbb{Q}, E[p^\infty] \otimes \chi)$$

We can study the different χ-parts independently.

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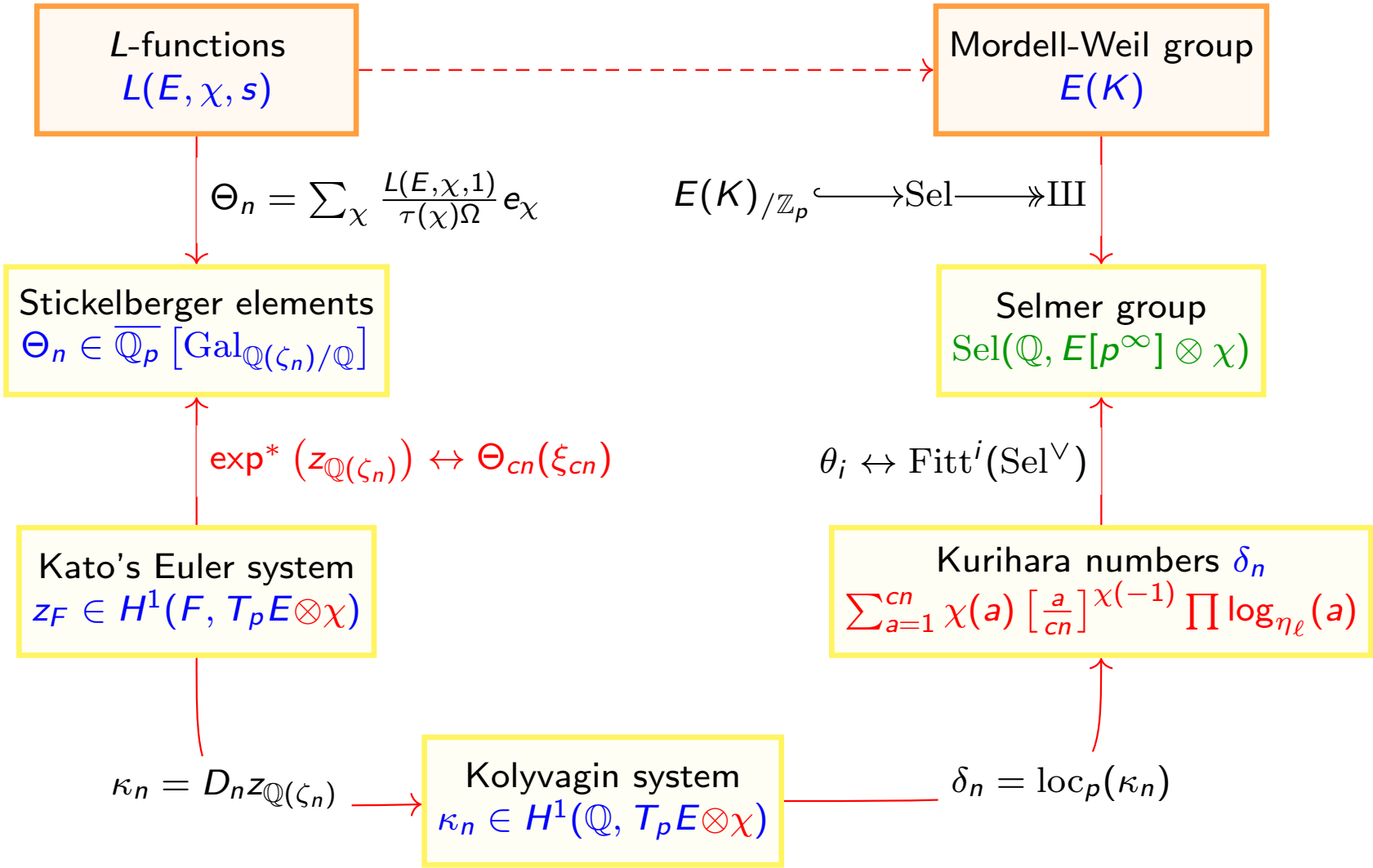

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- Problem: when c is not square-free, then ζ_{cn} do not generate $\mathbb{Q}(\zeta_{cn})$ as a $\mathbb{Q}_p[\mathcal{G}_{cn}]$-module.
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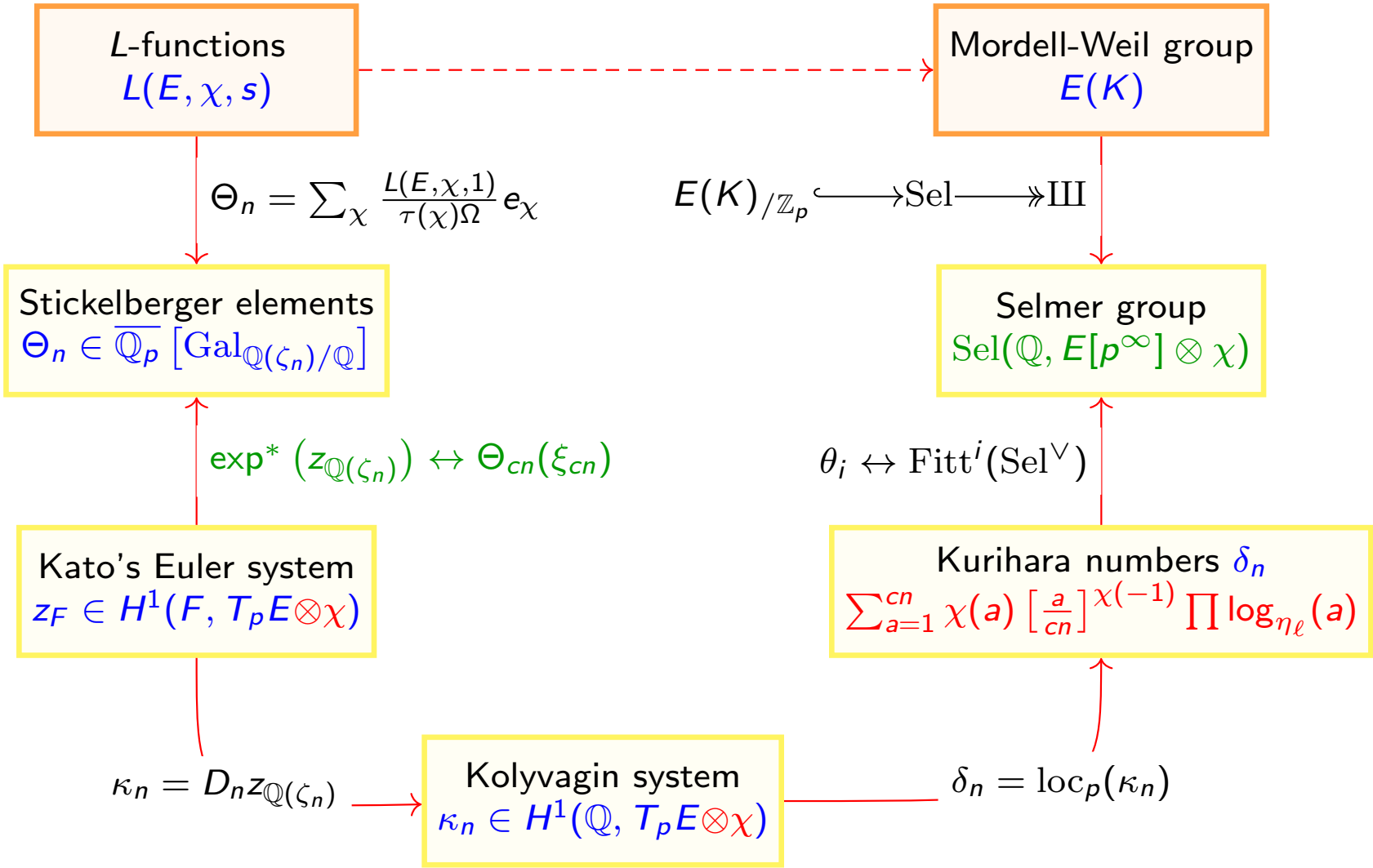

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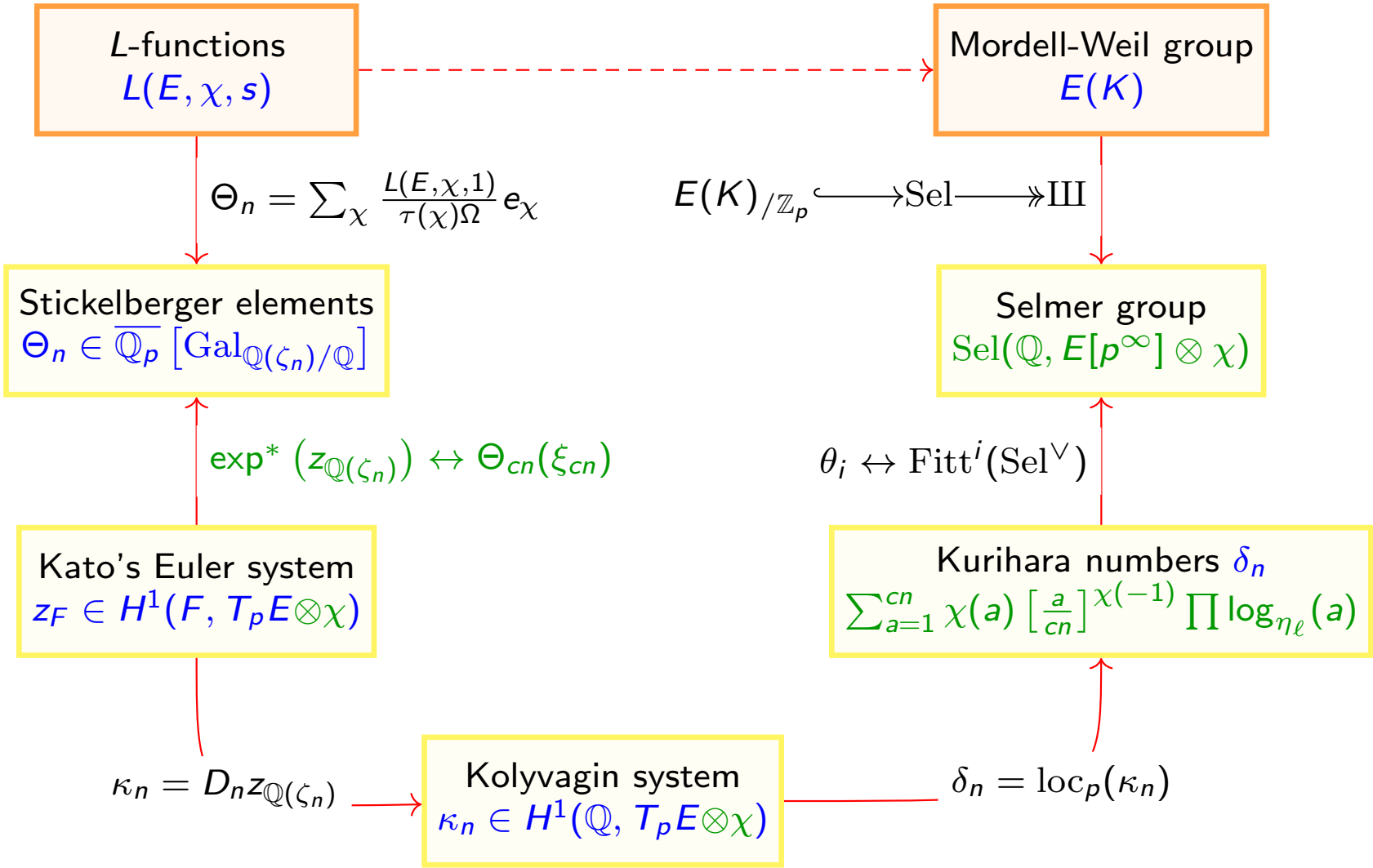
Twisted Kurihara numbers

Twisted Kato's Euler system $\rightarrow$ twisted Kurihara numbers

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/cn)^*} \chi(a) \left[\frac{a}{n}\right]^{\chi(-1)} \prod \left(\log_{\eta_\ell}(a)\right)$$

Arithmetic over an abelian extension $K/\mathbb{Q}$



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Thank you for your attention!