Kurihara numbers over abelian extensions

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- BSD relates the leading term of the L-function of an elliptic curve $E/\mathbb{Q}$ to the rank and the order of Tate-Shafarevich.
- Kurihara numbers are some quantities related to L-values that determine the structure of Tate-Shafarevich.
- Kurihara numbers are defined from the modular symbols of the elliptic curve.

$$\left[\frac{a}{n}\right] = 2\pi i \int_{i\infty}^{\frac{a}{n}} f(z)\,dz, \qquad \left[\frac{a}{n}\right]^{\pm} = \frac{1}{2\Omega_E^{\pm}}\left(\left[\frac{a}{n}\right] \pm \left[\frac{-a}{n}\right]\right) \in \mathbb{Q} \in \mathbb{Z}_p$$

- We will generalise the theory to describe the structure of $\mathrm{Sel}(K, E[p^\infty])$ as a $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$ for certain abelian extensions $K/\mathbb{Q}$.

Assumptions

Choose an elliptic curve $E/\mathbb{Q}$ and a prime number $p \geq 5$ satisfying the following

- (E1) The Manin constant of E is prime to p.
- (E2) The Galois group $G_{\mathbb{Q}}$ acts surjectively on T_pE.
- (E3) $E(\mathbb{Q}_p)$ contains no p-torsion.
- (E4) All Tamagawa numbers of E are prime to p.
- (E5) E satisfies the Iwasawa main conjecture.
- (E6) Ш is finite.

Definition

- For every $k \in \mathbb{N}$, consider the set of prime numbers $\mathcal{P}_k$ such that

$$\ell \equiv 1 \mod p^k, \qquad a_\ell \equiv \ell + 1 \mod p^k$$

- Note that $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \cdots \supset \mathcal{P}_k \supset \cdots$
- Denote by $\mathcal{N}_k$ the set of square free products of primes in $\mathcal{P}_k$.
- For every $n \in \mathcal{N}_1$, let k_n be the minimum $k \in \mathbb{N}$ such that $n \in \mathcal{N}_k$.

Kurihara numbers

$$\delta_n = \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^*} \left(\left[\frac{a}{n}\right]^+ + \left[\frac{a}{n}\right]^- \right) \prod_{\ell | n} \log_{\eta_\ell}(a) \in \mathbb{Z}/p^{k_n}$$

$$\boxed{\delta_1 = \frac{L(E,0)}{\Omega_E^+}}$$

Definition

$$\mathrm{ord}(\delta_n) = \max\left\{ j \in \mathbb{N} \cup \{0, \infty\} : \delta_n \in p^j \left(\mathbb{Z}/p^{k_n}\right) \right\}$$

Note that $\mathrm{ord}(\delta_n)$ is either less than k_n or ∞.

Rank and structure of the Tate-Shafarevich group

Definition

$$\alpha_i = \min\{\mathrm{ord}(\delta_n) : \ n \in \mathcal{N}_1, \ \nu(n) = i\} \in \mathbb{N} \cup \{0, \infty\}$$

Definition

- r is the minimum i such that α_i is finite.
- s is the minimum i such that α_i is zero.

Theorem

- *r is the rank of $E(\mathbb{Q})$.*
- $$\text{Ш} = \left(\frac{\mathbb{Z}}{p^{\frac{\alpha_r - \alpha_{(r+2)}}{2}}}\right)^2 \times \left(\frac{\mathbb{Z}}{p^{\frac{\alpha_{(r+2)} - \alpha_{(r+4)}}{2}}}\right)^2 \times \cdots \times \left(\frac{\mathbb{Z}}{p^{\frac{\alpha_{(s-2)} - \alpha_s}{2}}}\right)^2$$

Example (Cremona label 321398d1)

$$\boxed{E:\; y^2 + xy = x^3 - x^2 - 1\,531\,069\,681\,x - 23\,060\,083\,371\,235}$$

$$\mathrm{ord}_5(\delta_1) = \mathrm{ord}_5\left(\frac{L(E,1)}{\Omega_E^+}\right) = 4 \Rightarrow \quad \boxed{\mathrm{rank}(E) = 0, \quad \#\text{Ш} = 625 = 5^4}$$

Two possibilities

$$\text{Ш} \approx \mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/5 \quad \text{or} \quad \boxed{\text{Ш} \approx \mathbb{Z}/25 \times \mathbb{Z}/25}$$

One can check that $\ell_1 = 191$ and $\ell_2 = 401$ are in $\mathcal{P}_1$.
We compute $\mathrm{ord}_5(\delta_{\ell_1\ell_2}) = 0$.

Assume $E/\mathbb{Q}$ and $p \geq 5$ satisfy the following

- (E1) The Manin constant of E is prime to p.
- (E2) The Galois group $G_{\mathbb{Q}}$ acts surjectively on T_pE.
- (E3) $E(\mathbb{Q}_p)$ contains no p-torsion.
- (E4) All Tamagawa numbers of E are prime to p.

Theorem (Kim, Sakamoto)

The Iwasawa main conjecture is equivalent to the existence of some $n \in \mathcal{N}_1$ such that $\mathrm{ord}(\delta_n) = 0$.

Assume $(E/\mathbb{Q}, p)$ satisfies (E1)-(E6). Let $K/\mathbb{Q}$ be an abelian extension satisfying the following

- (K1) $d = [K : \mathbb{Q}]$ is prime to p.
- (K2) $K/\mathbb{Q}$ is unramified at p and at every bad prime of E.
- (K3) $E(K_{\mathfrak{p}})$ contains no p-torsion for every $\mathfrak{p}$ above p.
- (K4) All the Tamagawa numbers of E over K are prime to p.
- (K5) The Iwasawa main conjecture holds true for every twist of T_pE by a character of $G = \mathrm{Gal}(K/\mathbb{Q})$
- (K6) $Ш(E/K)$ is finite.

Remark

$$\mathrm{Sel}(K, E[p^\infty]) \sim \bigoplus_{\chi \in \hat{G}} \mathrm{Sel}(\mathbb{Q}, E[p^\infty] \otimes \chi)$$

Twisted Kurihara numbers

Twisted Kurihara numbers

$$\delta_{n,\chi} = \sum_{a \in (\mathbb{Z}/cn\mathbb{Z})^*} \chi(a) \left(\left[\frac{a}{cn} \right]^+ + \left[\frac{a}{cn} \right]^- \right) \prod_{\ell | n} \log_{\eta_\ell}(a) \in \mathbb{Z}/p^{k_n}$$

Definition

$$\alpha_{i,\chi} = \min\{\mathrm{ord}(\delta_{n,\chi}) : \; n \in \mathcal{N}_1, \, \nu(n) = i\}$$

Definition

- r_χ is the minimum i such that $\alpha_{i,\chi}$ is finite.
- s_χ is the minimum i such that $\alpha_{i,\chi}$ is zero.

Structure of the twisted Tate-Shafarevich groups

Case I: $\chi = \overline{\chi}$

- r_χ is the rank of $E(K)_\chi$.
- $Ш(E/K)_\chi = \left(\frac{O}{p^{\frac{\alpha_r - \alpha_{(r+2)}}{2}}} \right)^2 \times \cdots \times \left(\frac{O}{p^{\frac{\alpha_{(s-2)} - \alpha_s}{2}}} \right)^2$

Case II: $\chi \neq \overline{\chi}$

- r_χ is the rank of $E(K)_\chi$.
- $Ш(E/K)_\chi = \frac{O}{p^{\alpha_r - \alpha_{(r+1)}}} \times \cdots \times \frac{O}{p^{\alpha_{(s-1)} - \alpha_s}}$

Example (Cremona label 321398d1)

$$E: \ y^2 + xy = x^3 - x^2 - 1\,531\,069\,681\,x - 23\,060\,083\,371\,235$$

	Rank	χ	$\#Ш$	New characters
$\mathbb{Q}(\mu_{19})$	0	-	p^{10}	$\mathbb{F}_p$ in characters of order 9
$\mathbb{Q}(\mu_{43})$	0	-	p^4	-
$\mathbb{Q}(\mu_{79})$	1	Quadratic	p^8	$\mathbb{F}_p$ in characters of orders 3 or 6
$\mathbb{Q}(\mu_{83})$	0	-	p^4	-

Example

$$Ш(E/\mathbb{Q}(\mu_{19})) = \mathbb{Z}_p/p^2 \oplus \mathbb{Z}_p/p^2 \oplus \left(\frac{1}{3}(\sigma_1 + \sigma_{-1}) - \frac{1}{6}(\sigma_7 + \sigma_{-7} + \sigma_8 + \sigma_{-8})\right) \mathbb{F}_p[\mathrm{Gal}(\mathbb{Q}(\mu_{19})/\mathbb{Q})]$$

Thank you for your attention!