Kolyvagin systems and higher Fitting ideals of Selmer groups

Alberto Angurel Andrés

20489486

Thesis submitted to the University of Nottingham
for the degree of

Doctor of Philosophy

Supervised by

Dr. Christian Wuthrich

School of Mathematical Sciences

March 2026

Abstract

This thesis contributes to number theory and arithmetic geometry, specifically within the framework of Iwasawa theory. It advances the theory of Euler and Kolyvagin systems to determine the structure of Selmer groups in previously inaccessible cases: p-adic Selmer groups of rank 0 and Iwasawa Selmer groups of higher rank.

The computation in the rank 0 case is achieved by considering a Kolyvagin system for an auxiliary Selmer structure of rank one, and showing that its localisation determines most of the Fitting ideals, enough to describe the Selmer group up to isomorphism. As an application, we developed an algorithmic description of the classical Selmer group of an elliptic curve in terms of the modular symbols.

Secondly, we address Iwasawa Selmer modules using the theory of ultra Kolyvagin systems. These objects live in patched Galois cohomology groups, constructed using the set-theoretic notion of an ultrafilter to cluster countably many cohomology groups. We develop the theory of ultra Kolyvagin systems and apply it to compute Iwasawa Selmer groups up to pseudo-isomorphism.

The final part of this thesis introduces cartesian systems, a framework that generalises the existing theories of Kolyvagin and Stark systems. This setting leads to a formalism of Kolyvagin systems for Selmer structures of rank 0.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to Christian Wuthrich for his constant and invaluable support. This thesis would not have been possible without our long mathematical discussions; in addition to teaching me interesting mathematics, he offered me irreplaceable advice and guidance throughout these years.

I would also like to thank the School of Mathematical Sciences at the University of Nottingham for their support during this process, both financially and by providing an enjoyable research environment.

I am indebted to those who engaged in mathematical discussions regarding the topics of this thesis. In particular, I thank Dominik Bullach and David Loeffler, whose insight had a direct impact in the development of this work.

Finally, I would like to acknowledge my fellow Number Theory PhD students, Frederick and Lewis, who contributed to my mathematical knowledge with numerous study group talks on a wide variety of topics. Together with my other colleagues–Giovanni, Rory and Sam–they made my experience in Nottingham really enjoyable.

One of the best parts of this experience was having the opportunity to meet amazing people from all around the world. Some of them, Nacho, Pedro and others, always showed up at the different conferences I travelled to every summer, evolving from being my 'email spammers' to running up a mountain after a long day of mathematical talks.

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Chapter 1

Introduction

1.1 Historical Background

The method of Euler systems is a cornerstone of modern Iwasawa theory in which a collection of classes in Galois cohomology groups along a $\mathbb{Z}_p$-extension can be used to compute the structure of important arithmetic objects like class groups of number fields or the rank of the Mordell-Weil group of abelian varieties.

Euler systems connect these arithmetic objects with special values of L-functions, certain complex functions attached to these objects. This is one of the deepest connections in mathematics, relating an analytic object, the L-function, with some algebraic objects, such as class or Mordell-Weil groups. Furthermore, it can be also seen as a local-global principle, since global objects are determined in terms of local invariants, which are used to define the L-function. This connection is the origin of important conjectures such as those formulated by B. Birch and Sir H.P.F. Swinnerton-Dyer, H. Stark or S. Bloch and K. Kato

Euler systems are collections of cohomology classes living in different Galois cohomology groups and satisfying certain norm-compatibility relations among them. On the one hand, they can be related to the special values of the L-function via reciprocity laws and, on the other hand, they are the starting point of an argument that controls the structure of Selmer groups, which can be seen as a cohomological interpretation of certain arithmetic objects.

The ideas behind Euler systems were first used, independently, by F. Thaine [Tha88] and V.A. Kolyvagin in [Kol89] in order to bound, respectively, the class group of real abelian number fields and the rational points of some elliptic curves. It was in [Kol90] where V.A. Kolyvagin introduced the notion of Euler systems, considering a collection of points in the modular curves, known as Heegner points. He then applied a derivative operator to construct certain cohomology classes, now known as Kolyvagin derivatives. His work resulted in the proof of Birch and Swinnerton-Dyer conjecture when the analytic rank is one.

1.2 Kolyvagin systems

Later works [Per98], [Kat99] and [Rub00] generalised the theory to bound certain Selmer groups for arbitrary Galois representations. Those bounds are given by a collection of cohomology classes, called Kolyvagin's derivative classes, that can be obtained from every Euler system. In [MR04], B. Mazur and K. Rubin axiomatised the theory of these derivative classes to introduce the notion of Kolyvagin systems.

The theory of Kolyvagin systems works in great generality, allowing the study of Selmer groups defined for any Galois representation. We assume T is a Galois representation with coefficients in a local ring R, satisfying certain big image assumptions. A Selmer structure $\mathcal{F}$ on T is a collection of submodules of the local Galois cohomology which can be used to define Selmer groups $H^1_{\mathcal{F}}(K,T)$, which are the subgroups of the global cohomology group cut out by those local conditions (see Definitions 2.2.1 and 2.2.2).

Selmer groups control the structure of important arithmetic objects in number theory and arithmetic geometry. For example, the Bloch-Kato Selmer group, when defined for the Galois representation $T = \mathbb{Z}_p(1)$ measures the cardinality of class groups of number fields and, when defined for the Tate module of an abelian variety, controls the rank of the Mordell-Weil group.

Kolyvagin systems are collections of cohomology classes associated with a Selmer structure $\mathcal{F}$. More precisely, a Kolyvagin system κ is defined as

$$\kappa = \{\kappa_n\}_{n\in\mathcal{N}} \in \prod_{n\in\mathcal{N}} \mathbf{H}^1_{\mathcal{F}(n)}(K,T)$$

satisfying certain relations between them (see Definition 3.1.2). Here $\mathcal{N}$ is the set of square-free products of certain well-behaved primes, and $\mathcal{F}(n)$ is the Selmer structure obtained after suitably modifying the local conditions at the primes dividing n. The imposed compatibility relations are very restrictive so one can show, under certain technical assumptions, that the set of Kolyvagin systems, denoted by $\mathrm{KS}(\mathcal{F})$, is a free, cyclic module whose generators, called primitive Kolyvagin systems, carry information about the structure of the Selmer group.

The theory in [Rub00] and [MR04] uses the ideas from V.A. Kolyvagin to construct a map from the Euler systems to the Kolyvagin systems of a specific Selmer structure $\mathcal{F}^{\mathrm{can}}$. More precisely, it defines a map

$$\Psi: \ \mathrm{ES}(T) \to \mathrm{KS}(T, \mathcal{F}^{\mathrm{can}})$$

satisfying that, if $\Psi(\mathbf{c}) = \kappa$, then $c_{\mathbb{Q}} = \kappa_1$.

The Kolyvagin system argument depends on an invariant associated with the Selmer structure, the core rank $\chi(\mathcal{F})$, which measures the behaviour of the Selmer groups under duality pairings. The theory in [MR04] computes the Selmer group when $\chi(\mathcal{F}) = 1$ and R is a principal artinian ring. In loc. cit., a sequence of ideals $\Theta_i(\kappa)$ is associated with a Kolyvagin system κ (see Definition 3.1.8). They are constructed from the p-divisiblity of the classes κ_n, where n is the square-free product of exactly i primes.

When κ is a primitive Kolyvagin system and R is a principal artinian ring, the following equality holds for all $i \in \mathbb{Z}_{\geq 0}$:

$$\Theta_i(\kappa) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K, T^*)^\vee\big) \tag{1.1}$$

where $\mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K, T^*)^\vee\big)$ represents the i^{th} Fitting ideal of the Selmer group (see Definition B.2.1). Recall that the Fitting ideals are a sequence of ideals (see Definition B.2.1) that measure the full structure of R-modules and, in this case, determine the module up to isomorphism.

1.2.1 Kolyvagin systems of higher rank

The scope of the theory in [MR04] is limited to the case $\chi(\mathcal{F}) = 1$. The reason is that, when the core rank is higher, the module of Kolyvagin systems is bigger and cannot be generated by one element while, when the core rank is zero, there are no non-zero Kolyvagin systems. B. Mazur and K. Rubin came up with a solution in [MR16] that redefines Kolyvagin systems as collection of elements in the exterior powers of Selmer groups

$$\kappa = \{\kappa_n\}_{n\in\mathcal{N}} \in \prod_{n\in\mathcal{N}} \left(\bigwedge^{\chi(\mathcal{F})} H^1_{\mathcal{F}(\mathfrak{n})}(K, T) \right)$$

That solves the problem when R is a local, artinian, principal ring since the module of Kolyvagin systems becomes free of rank one. For every Kolyvagin system κ, one can construct a sequence of ideals $\Theta_i(\kappa)$ that, when κ is primitive, control the Fitting ideals of the Selmer group as in (1.1).

However, the fact that R is a principal ring is an essential condition in [MR16], which has been relaxed in the work of R. Sakamoto [Sak18] and D. Burns and T. Sano [BS21] and their subsequent joint paper [BSS25] to all self-injective noetherian rings R (see §B.1). The key innovation of that work is to consider Kolyvagin systems that are collection of elements in the exterior biduals (see Definition B.4.5) of Selmer groups:

$$\kappa = \{\kappa_n\}_{n\in\mathcal{N}} \in \prod_{n\in\mathcal{N}} \left(\bigcap^{\chi(\mathcal{F})} H^1_{\mathcal{F}(\mathfrak{n})}(K,T) \right)$$

This new definition is necessary to ensure that the module of Kolyvagin systems is a free module of rank one, even when R is not a principal ring.

The computation of the higher Fitting ideals of Selmer groups can be done using a new collections of classes, called Stark systems and denoted by $\mathrm{SS}(\mathcal{F})$, that are related to Kolyvagin systems using a regulator map

$$\mathrm{Reg}:\ \mathrm{SS}(\mathcal{F}) \to \mathrm{KS}(\mathcal{F})$$

Under mild assumptions, this regulator can be shown to be an isomorphism, so primitive Stark and Kolyvagin systems are related. It is also possible to extend the definition of the theta ideals to construct a sequence of ideals $\Theta_i(\varepsilon)$, associated with primitive Stark systems, satisfying that

$$\Theta_i(\kappa) \subset \Theta_i(\varepsilon) = \mathrm{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$$

Although the composition of the Kolyvagin derivative with the inverse map of the regulator produces a Stark system from an Euler system, this computation is far from explicit. Then it would be more desirable to give a computation of the Fitting ideals using the ideals $\Theta_i(\kappa)$ instead, i.e., to show the (conjectural) equality

$$\Theta_i(\kappa) = \Theta_i(\varepsilon)$$

Such an equality has only been proven when $i = 0$ or when the ring R is principal. We will approach this question in more generality using the formalism of ultra Kolyvagin systems described below.

1.2.2 Selmer groups of core rank zero

There are still some Selmer groups whose structure cannot be computed with the theory in [BSS25]. They are those coming from Selmer structures of core rank zero and include the important arithmetic examples like the classical Selmer group of an elliptic curve, which encodes information about the rank of Mordell-Weil group and plays a central role in the Birch and Swinnerton-Dyer conjecture. In this work, we develop a generalisation of the theory in [MR04], using Kolyvagin systems to compute Selmer groups of core rank zero up to isomorphism.

The main technical issue in this case is the lack of non-trivial Kolyvagin systems. Therefore, we are forced to consider an auxiliary Selmer structure $\mathcal{G}$ of core rank one, in which non-trivial Kolyvagin systems do exist. For each Kolyvagin system $\kappa \in \mathrm{KS}(\mathcal{G})$, we can construct theta ideals relative to $\mathcal{F}$ (see Definition 3.3.6), which will be denoted by $\Theta_i^{(0)}(\kappa, \mathcal{F})$, and that can be also related to the Fitting ideals of the Selmer group defined by $\mathcal{F}$.

Theorem 1.1 (Theorem 3.3.7)**.** *Let R be a principal, artinian, local ring with finite residue field. Let K be a number field and let T be an $R[[G_K]]$-module, unramified only at finitely many places, which is free and finitely generated as an R-module. Suppose that T satisfies a big image assumption, precisely stated in Assumption (BI). Assume $\mathcal{F} \leq \mathcal{G}$ (see Notation 3.3.2) are cartesian Selmer structures on T satisfying that $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. If $\kappa \in \mathrm{KS}(\mathcal{G})$ is a primitive Kolyvagin system, then*

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big) \ \forall i \in \mathbb{Z}_{\geq 0}$$

Moreover, if one of the following conditions is satisfied

(i) $i \leq \mathrm{rank}_R\big(H^1_{\mathcal{F}}(K,T)\big)$ (see Definition 3.3.8 below)

(ii) $\Theta_{i-1}^{(0)}(\kappa, \mathcal{F}) \subsetneq \mathrm{Fitt}_R^{i-1}\left(H^1_{\mathcal{F}}(K,T)\right)$

(iii) *The Fitting ideal $\mathrm{Fitt}_R^{i-1}\big(H^1_{\mathcal{F}}(K,T)\big)$ does not vanish and there is a strict inclusion*

$$\frac{\mathrm{Fitt}_R^{i-1}\big(H^1_{\mathcal{F}}(K,T)\big)}{\mathrm{Fitt}_R^{i}\big(H^1_{\mathcal{F}}(K,T)\big)} \subsetneq \frac{\mathrm{Fitt}_R^{i}\big(H^1_{\mathcal{F}}(K,T)\big)}{\mathrm{Fitt}_R^{i+1}\big(H^1_{\mathcal{F}}(K,T)\big)}$$

then the equality holds for the index i:

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

It is important to highlight that the second condition ensures that the equality between the ideal $\Theta_i^{(0)}(\kappa, \mathcal{F})$ and the i^{th} Fitting ideal of the Selmer groups holds for one index within each pair of consecutive ones.

There is one special case when the equality holds true for every index. That is the case when the Galois representation is not residually self-dual.

Theorem 1.2 (Theorem 3.4.1)**.** In addition to the assumptions of Theorem 1.1, assume that $T/\mathfrak{m}$ is not a self-dual Galois representation. If $\kappa \in \mathrm{KS}(\mathcal{G})$ is a primitive Kolyvagin system, then

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}}(K, T)) \ \forall i \in \mathbb{Z}_{\geq 0}$$

Theorems 1.1 and 1.2 can be used to compute Galois structure of the classical Selmer groups of elliptic curves defined over $\mathbb{Q}$, over certain abelian extensions (see Theorem 8.1.28). In this situation, the ideals $\Theta_i(\kappa)$ are constructed in terms of twisted Kurihara numbers $\widetilde{\delta}_n$, which are some analytic quantities, effectively computable, that can be defined from the modular symbols of the curve by the following formula (see Definition 8.1.22 for more details):

$$\widetilde{\delta}_{n,\chi} = \sum_{a \in (\mathbb{Z}/cn\mathbb{Z})^\times} \overline{\chi}(a) \left[\frac{a}{cn}\right]^{\chi(-1)} \left(\prod_{\ell \mid n} \log^p_{\eta_\ell}(a)\right)$$

1.3 Ultra Kolyvagin systems

1.3.1 Classical approach to infinite coefficients

All the Selmer groups previously mentioned are defined on finite Galois representations. However, interesting Selmer groups cannot be included in this group, since they arise from profinite Galois representations such as those with coefficients in p-adic rings or Iwasawa algebras.

The definition of Kolyvagin systems considers a collection of classes indexed by the set of square-free products $n \in \mathcal{N}$ of a set of well behaved primes, called Kolyvagin primes $\mathcal{P}$. All the primes in $\mathcal{P}$ share the property that the action of their Frobenius elements has eigenvalue 1 with single multiplicity. This property is an essential condition to define the Kolyvagin system relations.

The finiteness of the coefficient ring is a necessary requirement to guarantee the existence of Kolyvagin primes using the Chebotarev density theorem, together with a mild big image assumption.

This argument cannot be directly generalised when R is infinite, since the Chebotarev density theorem is only true for finite extensions. However, there are some attempts in [MR04] that aim to adapt the previous definitions of Kolyvagin systems to these representations, which are based on defining the Kolyvagin system relations on certain finite quotients of the Galois representation.

The first approach is to consider Kolyvagin systems $\kappa \in \mathrm{KS}(T, \mathcal{F})$ as collections of classes

$$\kappa = (\kappa_{\mathfrak{n}})_{n \in \mathcal{N}} \in \prod_{n \in \mathcal{N}} H^1_{\mathcal{F}(n)}(K, T \otimes R/I_n)$$

where I_n is an ideal of R depending on n. Then the Kolyvagin system relations are defined in the cohomology groups of certain finite quotients of T.

Another approach, which was also used in the higher rank theory in [BSS25] starts by expressing the coefficient ring as an inverse limit

$$R = \varprojlim_{k \in \mathbb{N}} R_k$$

where R_n are finite, self-injective rings, and consider elements in the inverse limit

$$\overline{\mathrm{KS}}(T, \mathcal{F}) = \varprojlim_{k \in \mathbb{N}} \mathrm{KS}(T_k, \mathcal{F})$$

Both approaches turn out to be equivalent when $\chi(\mathcal{F}) = 1$ and R is a discrete valuation ring (see [MR04, Theorem 5.2.9]).

1.3.2 Patched Cohomology

In this work, we offer a reinterpretation of $\overline{\mathrm{KS}}(T, \mathcal{F})$ as a direct Kolyvagin system construction using patched cohomology groups and ultraprimes introduced in [Swe22].

This process uses the set-theoretic notion of (non-principal) ultrafilters to construct a patching process that clusters countably many cohomology groups into a new group. This process leads to the construction of global and local patched cohomology, which also satisfies most properties expected for a cohomology theory, like the existence of a long exact sequence, duality pairings or inflation-restriction sequences.

Ultrafilters had been also used in other areas of mathematics. Arguably, their most important application appears in Łoś theorem [Łoś55], that considers the ultraproduct of logical models and can be used to prove the Compactness theorem, which had been originally proved by K. Gödel and A. Maltsev. Another important application is the non-standard analysis developed by A. Robinson in [Rob66], in which an ultrafilter is used to construct the set of hyperreal numbers, which contains infinitesimal (resp. infinite) numbers, i.e., numbers that are smaller (resp. larger) than all positive real numbers.

The main advantage of using ultrafilters in the theory of Kolyvagin systems is the generalization of the concept of primes to the notion of ultraprimes. Ultraprimes are constructed as equivalence classes in the set of sequences of prime numbers and each one of them comes with a notion of a local Galois group and local Galois cohomology. Their relation with the similarly constructed global patched cohomology groups leads to the construction of patched Selmer structures and patched Selmer groups. It is important to highlight that the classical theory is included in the patched theory, since it can be recovered when only constant ultraprimes[1] are considered, but the existence of non-constant ultraprimes is a powerful tool for the theory of Kolyvagin systems.

A generalisation of the Chebotarev density theorem for infinite extensions holds when you consider ultraprimes, instead of primes, which leads to the existence of Kolyvagin ultraprimes even when R is a profinite ring, assuming certain big image assumptions.

There is also a patched interpretation of the modified Selmer group $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)$, so we can define an ultra Kolyvagin systems

$$\kappa \in \mathbf{KS}(T,\mathcal{F})$$

as a collection of elements indexed by the square-free product of Kolyvagin ultraprimes $\mathcal{N}$:

$$\kappa = (\kappa_{\mathfrak{n}})_{n\in\mathcal{N}} \in \prod_{\mathfrak{n}\in\mathcal{N}} \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T) \right)$$

satisfying similarly defined Kolyvagin system relations.

In fact, ultra Kolyvagin systems can be obtained as limits of classical Kolyvagin systems, but this new interpretation allows to develop new relations with the Fitting ideals of Selmer groups.

[1] A constant ultraprime is the equivalence class of the sequence in which all the entries are the same prime number.

Theorem 1.3 (Theorem 5.2.19)**.** If R, T satisfy some technical hypothesis (see Assumptions (R-lim) and (BI)) and $\mathcal{F}$ is a cartesian Selmer structure (see Definition 5.1.5), there is a canonical isomorphism of free modules of rank one:

$$\mathbf{KS}(T,\mathcal{F}) = \overline{\mathrm{KS}}(T,\mathcal{F})$$

The construction of the theta ideals can be generalised to the formalism of ultra Kolyvagin systems, so there is a sequence of ideals $\boldsymbol{\Theta}_i(\kappa) \subset R$ associated with every $\kappa \in \mathbf{KS}(\mathcal{F})$. Similarly, these theta ideals can be obtained as limits of theta ideals of classical Kolyvagin systems.

Theorem 1.4 (Theorem 5.2.30)**.** Let $\kappa \in \mathbf{KS}(T,\mathcal{F})$ be an ultra Kolyvagin system and let $\overline{\kappa} = \left(\kappa^{(k)}\right)_{k\in\mathbb{N}} \in \overline{\mathrm{KS}}(T,\mathcal{F})$ be the image of κ under the identification in Theorem 1.3. Then, for every $i \in \mathbb{Z}_{\geq 0}$, we can recover the theta ideals of κ as

$$\boldsymbol{\Theta}_i(\kappa) = \varprojlim_{k\in\mathbb{N}} \Theta_i(\kappa)$$

When the core rank $\chi(\mathcal{F})$ (see Definition 2.2.14) is positive, we can look at the theta ideals $\Theta_i(\kappa)$ to establish bounds on the Selmer groups. Combining Theorem 1.4 with [BSS25, Theorem 5.2], we can obtain the following theorem.

Theorem 1.5 (Corollary 5.2.31)**.** Let $\mathcal{F}$ be a cartesian Selmer structure of positive core rank. For every ultra Kolyvagin system $\kappa \in \mathbf{KS}(T,\mathcal{F})$, its theta ideals are included in the Fitting ideals of the Selmer group:

$$\boldsymbol{\Theta}_i(\kappa) \subset \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$$

Moreover, if κ is primitive, the equality holds for $i=0$:

$$\boldsymbol{\Theta}_0(\kappa) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$$

It is an interesting question to see whether the equality holds for higher values of $i \in \mathbb{Z}_{\geq 0}$, whenever κ is a primitive Kolyvagin system. In case R is a discrete valuation ring, it follows from the classical theory in [MR04].

However, such equality is not known for classical Kolyvagin systems with more complicated coefficient rings, since the proof in [MR04] strongly uses the structure theorem of finitely generated modules over a principal, artinian, local ring. Therefore, the limit argument cannot be used when R is not a discrete valuation ring.

The formalism of ultra Kolyvagin systems contributes to link the theta ideals of the primitive ultra Kolyvagin systems with the Fitting ideals of the Selmer groups. In this work, we will prove they are equal up to finite index when R is the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[X]]$.

1.3.3 Discrete valuation rings

When R is a discrete valuation ring, ultra Kolyvagin systems determine completely the Fitting ideals of Selmer groups. Although it is possible to develop a theory of ultra Kolyvagin systems in characteristic zero, the main theorems in this section will follow by the limit argument and the results about classical Kolyvagin system in [MR04] and Theorems 1.1 and 1.2. Below we state the main results concerning the relation between Fitting ideals of Selmer groups and theta ideals of ultra Kolyvagin systems.

Theorem 1.6 (Theorem 5.3.9)**.** Let $\mathcal{F}$ be a cartesian Selmer structure of positive core rank $\chi(\mathcal{F}) > 0$ and let $\kappa \in \mathbf{KS}(T, \mathcal{F})$ be a primitive ultra Kolyvagin system. Then

$$\mathbf{\Theta}_i(\kappa) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) \ \forall i \in \mathbb{Z}_{\geq 0}$$

There is also a characterisation of the Selmer groups of core rank 0 using a generalisation of the construction of the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$, appearing in Theorems 1.1 and 1.2, to ultra Kolyvagin systems.

Theorem 1.7 (Theorem 5.4.6)**.** Assume that T is not a residually self-dual representation (satisfying Assumption (NSD)) and let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures of core rank $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. If $\kappa \in \mathbf{KS}(T, \mathcal{G})$ is a primitive ultra Kolyvagin system, then

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) \ \forall i \in \mathbb{Z}_{\geq 0}$$

There is also a comparison between the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$ and the Fitting ideals of the Selmer group when T is residually self-dual

Theorem 1.8 (Theorem 5.4.5)**.** Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$ and let $\kappa \in \mathrm{KS}(T)$ be a Kolyvagin system. Then

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) \ \forall i \in \mathbb{Z}_{\geq 0}$$

In addition, if κ is primitive and, for some fixed index $i \in \mathbb{Z}_{\geq 0}$, one of the following conditions is satisfied:

- $i \leq \operatorname{rank}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$.
- $\mathbf{\Theta}^{(0)}_{i-1}(\kappa, \mathcal{F}) \subsetneq \operatorname{Fitt}^{i-1}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$;
- The Fitting ideal $\operatorname{Fitt}^{i-1}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$ does not vanish and there is a strict inclusion

$$\frac{\operatorname{Fitt}^{i-1}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}{\operatorname{Fitt}^{i}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)} \subsetneq \frac{\operatorname{Fitt}^{i}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}{\operatorname{Fitt}^{i+1}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}$$

then the equality holds for the index i:

$$\mathbf{\Theta}^{(0)}_{i}(\kappa, \mathcal{F}) = \operatorname{Fitt}^{i}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

The conditions in Theorem 1.8 allow us to give a complete description of the Fitting ideals of the Selmer groups, providing a full description of its structure up to isomorphism.

Corollary 1.9 (Corollary 5.4.8)**.** Let $\mathcal{G}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{G}) = 1$ and let $\kappa \in \mathbf{KS}(T, \mathcal{F})$. Assume $\mathcal{F} \leq \mathcal{G}$ is another cartesian Selmer structure of core rank $\chi(\mathcal{F}) = 0$. Note that, for every $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we can find some $n_i \in \mathbb{Z}_{\geq 0}$ such that $\Theta^{(0)}_i(\widetilde{\kappa}, \mathcal{F}) = \mathfrak{m}^{n_i}$. Then all the Fitting ideals of the Selmer group can be computed as

$$\operatorname{Fitt}^{i}_R\left(H^1_{\mathcal{F}^*}(K, T^*)^\vee\right) = \mathfrak{m}^{\min\left\{n_i, \frac{n_{i+1}+n_{i-1}}{2}\right\}} \ \forall i \in \mathbb{Z}_{\geq 0}$$

1.3.4 Iwasawa algebra

It is for Selmer groups over more complicated rings when the ultra Kolyvagin system formalism can be used to compute Fitting ideals of Selmer groups.

In Theorem 1.5, it is proven that, when the core rank is positive, the theta ideals of a Kolyvagin systems are contained in the Fitting ideals of the Selmer group. For the index $i = 0$, this inclusion is, in fact, an equality.

It is mentioned in [BSS25, Remark 5.3] that it is an interesting question whether the equality extends to higher indices. That is proven in [MR04] when R is a principal ring, but the argument cannot be extended for more general self-injective rings.

In this section, we prove this question for certain quotients of the Iwasawa algebra. The innovation of our approach is that it uses an ultra Kolyvagin system formalism to study the Iwasawa Selmer group, using the full power of the structure theorem of finitely generated Iwasawa modules.

Then we are assuming that we are given a Galois representation $\mathbf{T}$ with coefficients in the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[X]]$ and develop an ultra Kolyvagin system formalism. When $\mathcal{F}$ is a cartesian Selmer structure whose core rank $\chi(\mathcal{F})$ is positive, the ultra Kolyvagin systems $\mathrm{KS}(\mathcal{F})$ form a free module of rank one and we can associate a sequence of ideals $\Theta_i(\kappa) \leq \Lambda$.

The formalism of ultra Kolyvagin systems over the Iwasawa algebra describes the Fitting ideals $\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)\big)$, which coincide with ideals $\Theta_i(\kappa)$ associated with a primitive $\kappa \in \mathbf{KS}(\mathbf{T}, \mathcal{F})$ and that this inclusion has finite index. This inclusion can be combined with the inclusion from Theorem 1.5.

Theorem 1.10 (Theorem 6.3.19, Corollary 6.3.20)**.** Let $\mathbf{T}$ be a Galois representation with coefficients in the Iwasawa algebra satisfying Assumptions (BI) and (H2) and let $\mathcal{F}$ be a cartesian Selmer structure whose propagation to $\mathbf{T}/(X)$ is also cartesian. If $\kappa \in \mathbf{KS}(\mathcal{F})$ is a primitive Kolyvagin system, then

$$\boldsymbol{\Theta}_i(\kappa) = \mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee\big)$$

1.4 Cartesian systems

The theory of higher rank Kolyvagin systems involve different collections of elements living in the exterior biduals of different Selmer groups. Depending on which Selmer structures we consider we have Kolyvagin or Stark systems.

Kolyvagin and Stark systems can be included in a more general kind of objects, known as cartesian systems. They are collections of elements in the exterior power of the Selmer groups associated with all possible (cartesian) Selmer structures that can be defined on T. More specifically, a cartesian system can be described, in a simplified way, as

$$\mathrm{CART}(\mathfrak{C}) := \varprojlim_{\mathcal{F} \in \mathfrak{C}} \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \right)$$

where the transition maps are defined, whenever $\mathcal{F} \leq \mathcal{G}$ (see Notation 3.3.2), using the rank reduction map from Propositions B.4.7 and B.4.8.

Since Kolyvagin and Stark systems are collections that are only described on a subset of the cartesian Selmer structures, they are examples of what are called partial cartesian systems. However, they can be uniquely extended to a (total) cartesian system.

In this thesis, we give a general theory that characterises under which conditions partial cartesian systems can be uniquely extended to total cartesian ones. If that is the case, for every cartesian Selmer structure $\mathcal{F}$, the element $\varepsilon_{\mathcal{F}}$ bounds the Fitting ideal of the Selmer groups associated with its dual Selmer structure $\mathcal{F}^*$. The key condition for showing that a partial cartesian system is extensible is that it is defined on a core Selmer structure $\mathcal{F}$, meaning that $\mathbf{H}^1_{\mathcal{F}^*}(K,T) = 0$.

When a partial cartesian system is not defined in any core Selmer structure, it might not be extended to a total cartesian system, but we may be able to prove that a certain scalar multiple of it can be. In this case, the elements $\varepsilon_{\mathcal{F}}$ establish weaker bounds on the Fitting ideals of Selmer groups.

Although it seems plausible to develop a theory of cartesian systems in the classical theory, we have opted to construct it using patched Selmer groups with profinite coefficient rings. There are two technical reasons behind this decision; the first one is the simplicity of cartesian Selmer structures, which implies that the intersection of two of them is still cartesian; the second, and most important, one is the notion of rank of modules over an integral domain and its additive behaviour on short exact sequences.

Cartesian systems provide a formalism to describe Kolyvagin systems for Selmer structures of rank 0. For those Selmer structures, the previous definitions of Kolyvagin systems cannot be used, since one cannot define the rank reduction maps required to establish the Kolyvagin system relations. From the theory of cartesian systems, there is a clear notion of Kolyvagin systems of rank 0, arising as subcollections of cartesian systems. It is an interesting problem trying to characterise them in an axiomatic way, similarly to the definition of Kolyvagin systems in higher rank.

The theory of cartesian systems aims to be a starting point for a Kolyvagin system argument under less restrictive Galois representation. It is expected that, under less restrictive hypotheses, Kolyvagin systems are partial cartesian systems that can only be extended to a total system after multiplying by a certain quantity, being able to establish (weaker) bounds on the Fitting ideals of the Selmer group. Another possible application is the existence of a unified theory of Kolyvagin systems, combining those obtained from cyclotomic and anticyclotomic Euler systems.

1.5 Structure of this thesis

Most parts of this thesis aim to compare the theta ideals associated with a (primitive) Kolyvagin system to the Fitting ideals of the Selmer group. This relation is studied when the coefficient ring R is self-injective, with the special case of principal artinian rings, a discrete valuation ring or the Iwasawa algebra. In all the cases except principal artinian rings or discrete valuation rings, the core rank $\chi(\mathcal{F})$ is assumed to be positive.

When $\chi(\mathcal{F})$ is positive, this relation is studied when R is a discrete valuation ring or the Iwasawa algebra, using ultra Kolyvagin systems. In the discrete valuation ring case, we recover the classical results in Chapter 5. When R is the Iwasawa algebra, the comparison between theta and Fitting ideals, which cannot be derived from the classical theory, is proven in Chapter 6.

When $\chi(\mathcal{F}) = 0$, this relation is only studied when R is a principal ring, either artinian or a discrete valuation ring. The artinian case is dealt in Chapter 3, while the generalisation to discrete valuation rings is shown in §5.4.

This thesis is structured in different parts. Chapters 2 and 3 are dedicated to the generalisation of the classical theory of Kolyvagin systems to compute the higher Fitting ideals when the core rank $\chi(\mathcal{F})$ is zero. This theory is a generalisation of the results in my preprint [Ang25, Theorems 1.1 and 1.3]. The theory included in these chapters is not needed in the subsequent parts of this section, with the exception of its direct generalisation in §5.4 and the application to elliptic curves in Chapter 8.

Chapter 4 is dedicated to the development of the patched cohomology formalism. The vast majority of it was already included in [Swe22]. This theory is used in Chapter 5 to develop a theory of ultra Kolyvagin systems in this formalism. The contents in Chapter 4 and 5 are used in subsequent Chapters 6 and 7 to build a Kolyvagin system theory for Iwasawa modules and the theory of cartesian systems. However, the author believes that Chapters 6 and 7 might be accessible without Chapters 4 and 5 if the reader is willing to treat the concept and existence of ultraprimes as a black box and treat patched cohomology as classical Galois cohomology.

As mentioned above, Chapter 6 is dedicated to the theory of ultra Kolyvagin systems for Iwasawa modules. This theory achieves the comparison between the theta ideals of a (primitive) ultra Kolyvagin system and the Fitting ideals of the (dual) Selmer group, which was not possible to obtain with the classical theory.

Chapter 7 is dedicated to the development of the theory of cartesian systems, which is a generalisation of the theory of Kolyvagin systems. The main applications of this theory are a unified theory of Kolyvagin systems or a formalism to construct Kolyvagin systems of when the core rank is zero.

Finally, the general theory has been applied Chapter 8 to provide an algorithmic description of the Galois structure of the classical Selmer group of an elliptic curve (defined over $\mathbb{Q}$) in terms of some analytic quantities, known as (twisted) Kurihara numbers, that can be defined in terms of the modular symbols of the curve. This algorithm has been implemented in Sagemath to numerically compute some specific examples (see §8.5). The contents of this chapter were already included in my preprint [Ang25, §4].

It is worth mentioning that this thesis includes two appendices. In Appendix A, we expose how to construct Kolyvagin systems using Euler systems. In Appendix B, we have included the algebraic background required in other parts of this thesis. In particular, the rank reduction maps in B.4.1 are central objects in the definitions of Kolyvagin systems.

Part I

Classical Kolyvagin systems

Chapter 2

Selmer structures

In this chapter, we present the required preliminaries about Selmer structures and Selmer groups, which will be required in Chapter 3 to develop the basic theory of Kolyvagin systems.

2.1 Kolyvagin primes

Notation 2.1.1. Let K be a number field. Fix an algebraic closure $\overline{K}$ of K. For every finite extension F/K, denote its absolute Galois group by $G_F = \mathrm{Gal}(\overline{K}/F)$.

We will consider Galois representations satisfying the following assumption.

Assumption (SI). Let R be a local, artinian and self-injective ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p (see B.1 for a description of the basic properties of self- injective rings). Let T be an $R[[G_K]]$-module, which is free and finitely generated as an R-module and is only ramified at finitely many primes.

In some results, the ring R will also be assumed to be a principal ring. When that is the case, it will be clearly stated.

Notation 2.1.2. (Duality) We will use the following duals of T

$$T^\vee := \mathrm{Hom}(T, \mathbb{Q}_p/\mathbb{Z}_p), \quad T^* := \mathrm{Hom}(T, \mu_{p^\infty}), \quad T^+ := \mathrm{Hom}(T, R)$$

Notation 2.1.3. We denote by $K(T)$ to the minimal Galois extension such that $G_{K(T)}$ acts trivially on T. Let M be the minimal power of p such that $MR = 0$ and let $K(1)$ be the maximal p-extension inside the Hilbert class field of K. Denote

$$K_M := K\big(\mu_M, (\mathcal{O}_K^\times)^{1/M}\big)K(1), \quad K(T)_M := K(T)K_M$$

We assume the following big image assumptions:

Assumption (BI). We assume the following assumptions:

- (T0) $p \geq 5$.
- (T1) $T/\mathfrak{m}T$ is an irreducible $k[[G_K]]$-module.
- (T2) There exists $\tau \in G_{K_M}$ such that $T/(\tau - 1)T \cong R$ as R-modules.
- (T3) $H^1(K(T)_M/K, T) = H^1(K(T)_M/K, T^*) = 0$.

Remark 2.1.4. Assume, in addition to (T0), that the homomorphism

$$\rho : \ G_{\mathbb{Q}} \to \mathrm{Aut}(T) \cong \mathrm{GL}_{\mathrm{rank}(T)}(R)$$

induced by the Galois action satisfies one of the following

- ρ is surjective.
- The image of ρ contains $\mathrm{SL}_{\mathrm{rank}(T)}(R)$ and $\mathrm{rank}(T) \geq 2$.

Then all three Assumptions (BI) hold. Indeed, (T1) and (T2) are clear.

For (T3), note that the order of $R^\times$ is divisible by $p-1$. It implies that the order of $\mathrm{GL}_{\mathrm{rank}(T)}(R)$ and, whenever $\mathrm{rank}(T) \geq 2$, also the order of $\mathrm{SL}_{\mathrm{rank}(R)}(T)$, are divisible by $p-1$.

Therefore, the order of $\mathrm{Gal}(K(T)_M/K)$, is also divisible by $p-1$. Then there is a subgroup $\Delta \subset \mathrm{Gal}(K(T)_M/K)$ of order $p-1$ such that, for every $A \in \{T, T^*\}$, $A^\Delta = 0$. By the inflation-restriction exact sequence

$$H^1\left(\frac{\mathrm{Gal}(K(T)_M/K)}{\Delta}, A^\Delta\right) \rightarrowtail H^1(K(T)_M/K, A) \longrightarrow H^1(\Delta, A)$$

Note that the first cohomology group vanishes since $A^\Delta = 0$ and the third one is also zero since the order of Δ is prime to p. Therefore, $H^1(K(T)_M/K, A)$ needs to be zero.

Remark 2.1.5. Assumptions (T1) and (T2) are the most restrictive ones, while (T3) is only a mild assumption. Indeed, it is shown in [LW17] that, when T is the Tate module of an elliptic curve, assumption (T3) in specific cases when $p \leq 11$, none of which satisfying all other assumptions.

Proposition 2.1.6. If T satisfies Assumption (BI), then, for every ideal I of R,

$$H^0(K, T/I) = H^0(K, T^*[I]) = 0$$

Proof. By (T1), either $(T/\mathfrak{m}T)^{G_K} = 0$ or $T/\mathfrak{m}T$ is the trivial representation. Since the latter does not satisfy (T3), then $(T/\mathfrak{m}T)^{G_K} = 0$. It implies that $(T/I)^{G_K} = 0$.

Similarly, (T1) and (T3) imply that $\big(T^*[\mathfrak{m}]\big)^{G_K} = 0$, so $H^0(K, T^*[I])$ vanishes for every ideal I. $\square$

There is a set of primes playing a crucial role in this theory.

Definition 2.1.7. A prime q is said to be a *Kolyvagin prime* if Frob_q is conjugate to τ in $\mathrm{Gal}(K(T)_M/K)$.

Notation 2.1.8. We define the following sets:

- $\mathcal{P}^{(R)}(T)$: set of Kolyvagin primes.
- $\mathcal{N}^{(R)}(T)$: set of square-free products of Kolyvagin primes.
- $\mathcal{N}_i^{(R)}(T)$: set of square-free products of i Kolyvagin primes.

When there is no risk of confusion, we will drop the reference to R and T.

The reason to choose these primes is that we can control its local cohomology, since the finite and singular cohomology groups, defined below, are free cyclic R-modules.

Definition 2.1.9 (Finite cohomology)**.** Let ℓ be a finite place of K, not dividing p. Assume T is unramified at ℓ. The *finite cohomology* group at ℓ is defined as

$$H^1_{\mathrm{f}}(K_\ell, T) := H^1(K_\ell^{\mathrm{ur}}, T) = \ker\Big(H^1(K_\ell, T) \to H^i(\mathcal{I}_\ell, T)\Big)$$

where K_ℓ^{ur}/K is the maximal unramified extension of K, $\mathcal{I}_\ell$ is the inertia subgroup of G_{K_ℓ}, and the second equality follows from the inflation-restriction sequence.

Remark 2.1.10. When the action of G_ℓ on T is ramified and $p \nmid \ell$, the local condition $H^1_{\mathrm{f}}(K_\ell, T)$ is defined only when T is obtained as the quotient of another Galois representation over a discrete valuation ring (see Definition 2.3.1). When $\ell \mid p$, the local condition $H^1_{\mathrm{f}}(K_\ell, T)$ usually refers to the Bloch-Kato local condition in Definition 2.3.3.

Definition 2.1.11 (Singular cohomology)**.** Let ℓ be a finite place of K as in Definition 2.1.9. The *singular cohomology* at ℓ is the

quotients
$$H^i_{\mathrm{s}}(K_\ell, T) = H^i(K_\ell, T) \Big/ H^i_{\mathrm{f}}(K_\ell, T)$$

When ℓ is a Kolyvagin prime, the singular cohomology can be also identified with a subgroup of $H^1(K_\ell, T)$.

Proposition 2.1.12. ([MR04, Lemma 1.2.1]) If $\ell \in \mathcal{P}$, the canonical short exact sequence

$$0 \longrightarrow H^1_{\mathrm{f}}(K_\ell, T) \longrightarrow H^1(K_\ell, T) \longrightarrow H^1_s(K_\ell, T) \longrightarrow 0 \tag{2.1}$$

splits canonically. Moreover, there exist isomorphisms of free cyclic R-modules

$$H^1_{\mathrm{f}}(K_\ell, T) = T/(\tau - 1)T, \quad H^1_{\mathrm{s}}(K_\ell, T) \cong T^{\tau=1}$$

Remark 2.1.13. The first isomorphism is canonical from the identification

$$H^1_{\mathrm{f}}(K_\ell, T) = T/(\mathrm{Frob}_\ell - 1)T = T/(\tau - 1)T$$

However, the second one is only canonical after tensoring with the Galois group
$$\mathcal{G}_\ell := \mathrm{Gal}(K(\ell)/K(1))$$
where $K(\ell)$ is defined as the maximal p-extension inside the ray class field modulo ℓ. Following [MR04, Lemma 1.2.1]:

$$H^1_{\mathrm{s}}(K_\ell, T) \otimes_{\mathbb{Z}} \mathcal{G}_\ell = \mathrm{Hom}(\mathcal{I}_\ell, T^{\mathrm{Frob}_\ell = 1}) \otimes \mathcal{G}_\ell = T^{\mathrm{Frob}_\ell = 1} = T^{\tau=1}$$

Definition 2.1.14 (Transverse cohomology)**.** Let $\ell \in \mathcal{P}$. The *transverse cohomology* subgroup is defined as the image of the inflation map

$$H^1_{\mathrm{tr}}(K_\ell, T) := H^1\big(K(\ell)_\ell/K_\ell, T^{G_{K(\ell)_\ell}}\big) \hookrightarrow H^1(K_\ell, T)$$

Proposition 2.1.15. ([MR04, Lemma 1.2.4]) $H^1_{\mathrm{tr}}(K_\ell, T)$ is the image of the canonical splitting in Equation (2.1). Note it is canonically isomorphic to $H^1_{\mathrm{s}}(K_\ell, T)$.

There is a canonical isomorphism between the finite and the singular cohomology at a Kolyvagin prime ℓ.

Proposition 2.1.16. ([MR04, Lemma 1.2.3]) Let $\ell \in \mathcal{P}$. Then there is a canonical isomorphism, known as *finite-singular map*,

$$\varphi_\ell^{\mathrm{fs}}: \; H^1_{\mathrm{f}}(K_\ell, T) \to H^1_{\mathrm{s}}(K_\ell, T) \otimes \mathcal{G}_\ell$$

Notation 2.1.17. In order to simplify notation, we fix once and for all, and for each Kolyvagin prime $\ell \in \mathcal{P}$, a generator τ_ℓ of $\mathcal{G}_\ell$. This choice, together with the finite singular map, fixes an isomorphism between $H^1_{\mathrm{f}}(K_\ell, T)$ and $H^1_{\mathrm{s}}(K_\ell, T)$.

The finite and transverse cohomology groups behave well under local Tate duality.

Proposition 2.1.18. (Local Tate duality, [Rub00, Theorem 1.4.1 and Proposition 1.4.2], [MR04, Proposition 1.3.2]) There is a perfect pairing

$$H^1(K_\ell, T) \times H^1(K_\ell, T^*) \to \mathbb{Q}_p/\mathbb{Z}_p$$

Under this pairing,

- $H^1_{\mathrm{f}}(K, T)$ and $H^1_{\mathrm{f}}(K, T^*)$ are annihilators of each other.
- If $\ell \in \mathcal{P}$, then $H^1_{\mathrm{tr}}(K, T)$ and $H^1_{\mathrm{tr}}(K, T^*)$ are annihilators of each other.

2.2 Selmer modules

In this section, we introduce the concepts of Selmer structures and their associated Selmer modules. They are subgroups of the global Galois cohomology which are cut out by local conditions. The can be used to determine the structure of important arithmetic objects like class groups of number fields or Mordell-Weil groups of abelian varieties.

Definition 2.2.1. A *Selmer structure* $\mathcal{F}$ on T is a collection of the following data:

- A finite set $\Sigma_{\mathcal{F}}$ of places of K, including all archimedean and p-adic primes and all the primes where T is ramified.
- For every $\ell \in \Sigma_{\mathcal{F}}$, a choice of an R-submodule

  $$H^1_{\mathcal{F}}(K_\ell, T) \subset H^1(K_\ell, T)$$

  This choice is known as *local condition* at ℓ.

Definition 2.2.2. The *Selmer module* associated with a Selmer structure is

$$H^1_{\mathcal{F}}(K,T) = \ker\left(H^1(K^\Sigma/K,T) \to \prod_{\ell\in\Sigma} H^1(K_\ell,T)\right)$$

where K^Σ/K is the maximal extension unramified outside Σ and the map is the composition of inflation and restriction map

Remark 2.2.3. When $\ell \notin \Sigma_{\mathcal{F}}$, we say the local condition at ℓ is

$$H^1_{\mathcal{F}}(K_\ell,T) = H^1_f(K_\ell,T)$$

Under this identification, the Selmer module only depends on the local conditions, and not on the set $\Sigma_{\mathcal{F}}$, being

$$H^1_{\mathcal{F}}(K,T) = \ker\left(H^1(K,T) \to \prod_{\ell\in\mathbb{P}_K} H^1(K_\ell,T)\right)$$

In order to compare Selmer structures, Poitou-Tate global duality is helpful. In order to introduce the global duality exact sequence, we need to introduce the concept of dual Selmer structure.

Definition 2.2.4 (Dual Selmer structure)**.** If $\mathcal{F}$ is a Selmer structure defined on T, there is a *dual Selmer structure* defined on T by the data

- $\Sigma_{\mathcal{F}^*} = \Sigma_{\mathcal{F}}$
- For $\ell \in \Sigma$, $H^1_{\mathcal{F}^*}(K_\ell,T)$ is defined to be the annihilator of $H^1_{\mathcal{F}}(K_\ell,T)$ under the pairing in Proposition 2.1.18.

There is a natural partial order relation in the set of Selmer structures

Definition 2.2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be Selmer structures. We say that $\mathcal{F} \leq \mathcal{G}$ if

$$H^1_{\mathcal{F}}(K_\ell,T) \subset H^1_{\mathcal{G}}(K_\ell,T)\ \forall \ell\in\mathbb{P}_K$$

Whenever two Selmer structures are ordered, their Selmer groups are related by the global duality exact sequence.

Proposition 2.2.6. ([MR04, Theorem 2.3.4]) Let $\mathcal{F} \leq \mathcal{G}$ be Selmer structures. Then the following sequence, where the third map is induced by Proposition 2.1.18, is exact.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^1_{\mathcal{F}}(K,T) \arrow[r] & H^1_{\mathcal{G}}(K,T) \arrow[r] & \bigoplus_{\ell\in\Sigma_{\mathcal{F}}\cup\Sigma_{\mathcal{G}}} \dfrac{H^1_{\mathcal{G}}(K_\ell,T)}{H^1_{\mathcal{F}}(K_\ell,T)} \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& H^1_{\mathcal{G}^*}(K,T^*)^\vee \arrow[r] & H^1_{\mathcal{F}^*}(K,T^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

Local conditions propagate naturally to submodules and quotients of T.

Definition 2.2.7 (Propagation to submodules)**.** Let $T' \hookrightarrow T$ be a submodule. This inclusion induces a map

$$\mu: \ H^1(K,T') \to H^1(K,T)$$

A local condition at T propagates to T' as

$$H^1_{\mathcal{F}}(K_\ell,T') := \mu^{-1}\big(H^1_{\mathcal{F}}(K_\ell,T)\big)$$

Definition 2.2.8 (Propagation to quotients)**.** Let $T \hookrightarrow T''$ be a quotient map. It a map

$$\varepsilon: \ H^1(K,T) \to H^1(K,T'')$$

A local condition at T propagates to T' as

$$H^1_{\mathcal{F}}(K_\ell,T'') := \varepsilon\big(H^1_{\mathcal{F}}(K_\ell,T)\big)$$

Remark 2.2.9. Let $T_1 \subset T_2 \subset T$ be two submodules of T. The propagation of a local condition to the subquotient T_2/T_1 is independent of the order in which we perform the operations.

With the definition of the propagation of Selmer structures, we can compare the Selmer groups of submodules with the torsion of the Selmer group.

Proposition 2.2.10. ([MR04, Lemma 3.5.3], [BSS25, Proposition 3.5]) Under Assumptions (BI), for every ideal I of R, the canonical inclusion $T[I] \hookrightarrow T$ induces an isomorphism

$$H^1_{\mathcal{F}}(K,T[I]) = H^1_{\mathcal{F}}(K,T)[I]$$

In this theory, it is required to impose a technical condition on the Selmer structures that guarantees good behaviour under the propagation.

Definition 2.2.11 (Cartesian Selmer structure)**.** A Selmer structure $\mathcal{F}$ is said to be *cartesian* if the map

$$H^1_{/\mathcal{F}}(K_\ell, T \otimes k) \to H^1_{/\mathcal{F}}(K_\ell, T)$$

is injective for every prime ℓ.

Remark 2.2.12. It is enough to check the cartesian condition for $\ell \in \Sigma_{\mathcal{F}}$. Indeed, when $\ell \notin \Sigma_{\mathcal{F}}$, then

$$H^1_{\mathcal{F}}(K_\ell, T) = H^1_{\mathrm{f}}(K_\ell, T)$$

Therefore,

$$H^1_{/\mathcal{F}}(K_\ell, T) = H^1_{\mathrm{s}}(K_\ell, T) = \mathrm{Hom}(\mathcal{I}_\ell, T^{\mathrm{Frob}_\ell = 1})$$

which is a cartesian local condition because Hom is a left exact functor.

When the Selmer structure is cartesian, the Selmer group of some quotients of T can be also identified with the torsion of the Selmer group.

Proposition 2.2.13. Suppose Assumptions (BI) hold and I is an ideal of R such that $R[I]$ is principal. The multiplication by a generator π induces an injection $T/I \hookrightarrow T$, which itself induces an isomorphism

$$H^1_{\mathcal{F}}(K, T/I) \hookrightarrow H^1_{\mathcal{F}}(K, T)[I]$$

Proof. Multiplication by π induces an isomorphism $T/I \cong T[I]$. Therefore, Proposition 2.2.10 implies that

$$H^1_{\mathcal{F}}(K, T/I) = H^1_{\mathcal{F}}(K, T[I]) = H^1_{\mathcal{F}}(K, T)[I] \qquad \square$$

The theory of Kolyvagin systems is dependent on the core rank, which is an invariant associated with the Selmer structure, that measures the difference in dimension between the Selmer module and the Selmer module of the dual structure.

Definition 2.2.14 (Core rank)**.** Let $\mathcal{F}$ be a Selmer structure on T. The *core rank* of $\mathcal{F}$ is the integer

$$\chi(\mathcal{F}) := \dim_k H^1_{\mathcal{F}}(K, T \otimes k) - \dim_k H^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}])$$

Remark 2.2.15. We will assume $\chi(\mathcal{F})$ is non-negative. Otherwise, one could swap the roles of $\mathcal{F}$ and $\mathcal{F}^*$ since $\chi(\mathcal{F}^*) = -\chi(\mathcal{F})$.

When the Selmer structure is cartesian, the core rank can determine the relation of the full Selmer group with the one of the dual Selmer structure.

Proposition 2.2.16. ([MR04, Theorem 4.1.5]) Let R be a principal ring satisfying Assumption (SI) and let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F}) \geq 0$. Then there is a non-canonical homomorphism

$$H^1_{\mathcal{F}}(K, T) = R^{\chi(\mathcal{F})} \oplus H^1_{\mathcal{F}^*}(K, T^*)$$

The argument to compute the structure of a Selmer group involves modifying the local conditions at certain primes. In order to do that, we will set the following definition.

Definition 2.2.17. Let $\mathcal{F}$ be a Selmer structure and let $a, b, c \in NN$ be pairwise coprime. Assume $c \in \mathcal{N}$. Define the Selmer structure $\mathcal{F}_a^b(c)$ by the local conditions

$$H^1_{\mathcal{F}_a^b(c)}(K_\ell, T) = \begin{cases} 0 & \text{if } \ell | a \\ H^1(K_\ell, T) & \text{if } \ell | b \\ H^1_{\mathrm{tr}}(K_\ell, T) & \text{if } \ell | c \\ H^1_{\mathcal{F}}(K_\ell, T) & \text{otherwise} \end{cases}$$

By Proposition 2.1.18, we can determine explicitly the dual of the modified Selmer structures.

Proposition 2.2.18. Let $\mathcal{F}$ be a cartesian Selmer structure and let $a, b, c \in \mathcal{N}$ be pairwise coprime. Then

$$\left(\mathcal{F}_a^b(c)\right)^* = (\mathcal{F}^*)_b^a(c)$$

We can relate the core rank of $\mathcal{F}_a^b(c)$ with the core rank of $\mathcal{F}$ and the number of prime divisors of a and b.

Notation 2.2.19. For every $n \in \mathcal{N}$, we denote by $\nu(n)$ the number of prime divisors of n.

Proposition 2.2.20. ([Sak18, Corollary 3.21]) Let $\mathcal{F}$ be a cartesian Selmer structure and let $a, b, c \in \mathcal{N}$ be pairwise coprime. Then $\mathcal{F}_a^b(c)$ is also cartesian and

$$\chi\big(\mathcal{F}_a^b(c)\big) = \chi(\mathcal{F}) + \nu(b) - \nu(a)$$

The Kolyvagin system argument involves the modification of certain conditions in order to make the Selmer module smaller. For this reason, we will finish this section with some technical lemmas that will be used repeatedly. We start with an application of the Chebotarev density theorem that proves the existence of Kolyvagin primes such that their localisation does not annihilate certain elements in the local cohomology group.

Proposition 2.2.21. ([BSS25, Lemma 3.9]) Consider non-zero cohomology classes

$$c_1, \ldots, c_s \in H^1(K, T), \quad c_1^*, \ldots, c_t^* \in H^1(K, T^*)$$

If $s + t < p$, there is a Kolyvagin prime $\ell \in \mathcal{P}$ such that $\mathrm{loc}_\ell(c_i)$ and $\mathrm{loc}_\ell(c_i^*)$ are all non-zero.

Lemma 2.2.22. ([MR04, Lemma 4.1.7]) Let $\mathcal{F}$ be a Selmer structure and let $\ell \in \mathcal{P}$ be a prime satisfying that

$$\mathrm{loc}_\ell : \ H^1_{\mathcal{F}}(K, T) \to H^1_{\mathrm{f}}(K_\ell, T)$$

is surjective. Then $H^1_{(\mathcal{F}^*)(\ell)}(K, T^*) = H^1_{(\mathcal{F}^*)_\ell}(K, T^*)$.

Proof. The surjectivity of loc_ℓ, together with the exact sequence of Proposition 2.2.6 with Selmer structures $\mathcal{F}_\ell$ and $\mathcal{F}$ implies that

$$H^1_{(\mathcal{F}^*)^\ell}(K, T^*) = H^1_{\mathcal{F}^*}(K, T^*) \tag{2.2}$$

By construction, we know that

$$H^1_{(\mathcal{F}^*)_\ell}(K, T^*) = H^1_{\mathcal{F}^*}(K, T^*) \cap H^1_{(\mathcal{F}^*)(\ell)}(K, T^*)$$

By (2.2)

$$H^1_{(\mathcal{F}^*)_\ell}(K, T^*) = H^1_{(\mathcal{F}^*)^\ell}(K, T^*) \cap H^1_{(\mathcal{F}^*)(\ell)}(K, T^*)$$

Since $H^1_{(\mathcal{F}^*)(\ell)}(K, T^*) \subset H^1_{(\mathcal{F}^*)^\ell}(K, T^*)$, we get that

$$H^1_{(\mathcal{F}^*)_\ell}(K, T^*) = H^1_{(\mathcal{F}^*)(\ell)}(K, T^*)$$

□

For the rest of this section, we assume R is a principal ring. The next two lemmas show how we can make the Selmer group smaller by swapping the local condition at a certain appropriate prime ℓ. The situation when $\chi(\mathcal{F}) \geq 1$ was done in [MR04].

Lemma 2.2.23. ([MR04, Proposition 4.5.8]) Suppose that R is a principal ring satisfying Assumption (SI) and that $\mathcal{F}$ be a cartesian Selmer structure. Assume that $H^1_{\mathcal{F}}(K,T)$ contains a submodule isomorphic to R and that

$$H^1_{\mathcal{F}^*}(K,T^*) \cong R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

for some exponents $e_1 \geq e_2 \geq \cdots \geq e_s$. Then there exists a Kolyvagin prime $\ell \in \mathcal{P}$ such that

$$H^1_{\mathcal{F}^*(\ell)}(K,T^*) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Remark 2.2.24. When $\chi(\mathcal{F}) \geq 1$, the Selmer group $H^1_{\mathcal{F}}(K,T)$ always contains a submodule isomorphic to R, since there is a non-canonical isomorphisms

$$H^1_{\mathcal{F}}(K,T) \cong R^{\chi(\mathcal{F})} \oplus H^1_{\mathcal{F}^*}(K,T^*)$$

Proof of Lemma 2.2.23. Let n be the length of R and let π be a generator of $\mathfrak{m}$. Choose classes $c \in H^1_{\mathcal{F}}(K,T)$ and $c^* \in H^1_{\mathcal{F}^*}(K,T^*)$ such that $\pi^{n-1}c \neq 0$ and $\pi^{e_1-1}c^* \neq 0$. By Proposition 2.2.21, there is a Kolyvagin prime $\ell \in \mathcal{P}$ such that

$$\mathrm{loc}_\ell(\pi^{n-1}c) \neq 0, \quad \mathrm{loc}_\ell(\pi^{e_1-1}c^*) \neq 0$$

The first condition implies that loc_ℓ is surjective, so Lemma 2.2.23 implies that

$$H^1_{(\mathcal{F}^*)(\ell)}(K,T^*) = H^1_{(\mathcal{F}^*)_\ell}(K,T^*) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s} \qquad \square$$

When $\chi(\mathcal{F}) = 0$, we need to study quotients of T in order to apply 2.2.22 and, when recovering the information about the Selmer group of T, we only get partial information. The next lemma is the technical base for the main Theorem 3.3.7 below about the structure of Selmer groups of core rank zero.

Lemma 2.2.25. Let R be a principal ring satisfying Assumption (SI) and let $\mathcal{F}$ be a cartesian Selmer structure such that $\chi(\mathcal{F}) = 0$. Assume that

$$H^1_{\mathcal{F}}(K,T) \cong R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

for some exponents $e_1 \geq e_2 \geq \cdots \geq e_s$. Then there exists a Kolyvagin prime $\ell \in \mathcal{P}$ and an integer t, such that $e_2 \leq t \leq k$.

$$H^1_{\mathcal{F}}(K,T) \cong R/\mathfrak{m}^{t} \times R/\mathfrak{m}^{e_3} \cdots \times R/\mathfrak{m}^{e_s}$$

If, moreover, $e_1 > e_2$, the integer t can be chosen equal to e_2.

Proof. Since $\chi(\mathcal{F}) = 0$, Proposition 2.2.16 implies that

$$H^1_{\mathcal{F}^*}(K,T^*) \cong H^1_{\mathcal{F}}(K,T) \cong R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

We pick classes $c \in H^1_{\mathcal{F}}(K,T)$ and $c^* \in H^1_{\mathcal{F}^*}(K,T^*)$ such that $\pi^{e_1}c \neq 0$ and $\pi^{e_1}c^* \neq 0$. By Proposition 2.2.21, we can choose a Kolyvagin prime $\ell \in \mathcal{P}$ such that

$$\mathrm{loc}_\ell(\pi^{e_1-1}c) \neq 0, \quad \mathrm{loc}_\ell(\pi^{e_1-1}c^*) \neq 0$$

Consider the diagram

$$\begin{array}{ccc} H^1_{\mathcal{F}}(K,T/\mathfrak{m}^{e_1}) & \xrightarrow{\mathrm{loc}_\ell} & H^1_{\mathrm{f}}(K_\ell,T/\mathfrak{m}^{e_1}) \\ \downarrow{\scriptstyle \pi^{k-e_1}} & & \downarrow{\scriptstyle \pi^{k-e_1}} \\ H^1_{\mathcal{F}}(K,T) & \xrightarrow{\mathrm{loc}_\ell} & H^1_{\mathrm{f}}(K_\ell,T) \end{array}$$

By Proposition 2.2.13, the leftmost vertical map is surjective, so there is some $c' \in H^1_{\mathcal{F}}(K,T/\mathfrak{m}^{e_1})$ such that $\mathrm{loc}_\ell(\pi^{e_1}c') \neq 0$.

Note that the element $\tau \in G_K$ from (T2) in Assumption (BI) satisfies that

$$T/(\mathfrak{m}^{e_1}, \tau - 1) \cong R/\mathfrak{m}^{e_1}$$

and, since $K(T/\mathfrak{m}^{e_1})_{p^{e_1}} \subset K(T)_{;}$, then Frob_ℓ is conjugate to τ in $\mathrm{Gal}(K(T/\mathfrak{m}^{e_1})_{p^{e_1}}/K)$, so it is a Kolyvagin prime for $T/\mathfrak{m}^{e_1}$. Then we can apply Lemma 2.2.22 to guarantee that

$$H^1_{(\mathcal{F}^*)(\ell)}(K,T^*[\mathfrak{m}^{e_1}]) = H^1_{(\mathcal{F}^*)_\ell}(K,T^*[\mathfrak{m}^{e_1}]) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

By Proposition 2.2.10,

$$H^1_{\mathcal{F}^*(\ell)}(\mathbb{Q},T^*)[\mathfrak{m}^{e_1}] = H^1_{\mathcal{F}^*(\ell)}(\mathbb{Q},T^*[\mathfrak{m}^{e_1}]) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Finally, Proposition 2.2.16 implies that

$$H^1_{\mathcal{F}(\ell)}(K,T)[\mathfrak{m}^{e_1}] \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Since $\chi(\mathcal{F}^\ell) = 1$ by Proposition 2.2.20, then Proposition 2.2.16 implies that

$$H^1_{\mathcal{F}(\ell)}(\mathbb{Q},T) \subset H^1_{\mathcal{F}^\ell}(\mathbb{Q},T) \cong R \oplus H^1_{\mathcal{F}^*_\ell}(\mathbb{Q},T^*)$$

Therefore, $H^1_{\mathcal{F}(\ell)}(\mathbb{Q},T)$ can be injected into $R \times R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$ and its $\mathfrak{m}^{e_1}$-torsion is isomorphic to $R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$. Under those considerations, the lemma follows by the structure theorem of $R/\mathfrak{m}^k$-modules. □

Even in the case $\chi(\mathcal{F}) = 0$, we can improve Lemma 2.2.25 to obtain a result of the kind of Lemma 2.2.23 even when the Selmer group does not contain submodules isomorphic to R, but assuming some hypothesis about T not being residually self-dual.

Assumption (NSD). Consider the following assumptions to rule out self-duality in T

- (N1) $T/\mathfrak{m}T$ is not isomorphic to $T^*[\mathfrak{m}]$ as $k[[G_K]]$-modules.
- (N2) The image of the homomorphism $R \to \mathrm{End}(T)$ is contained in the image of $\mathbb{Z}_p[[G_\mathbb{Q}]] \to \mathrm{End}(T)$.

Under those assumptions, we have a stronger application of the Chebotarev density theorem.

Proposition 2.2.26. ([MR04, Proposition 3.6.2]) Assume that T satisfies Assumptions (BI) and (NSD). Let $C \subset H^1(K,T)$ and $D \subset H^1(K,T^*)$ be finite submodules and choose homomorphisms

$$\phi:\ C \to R, \quad \psi: D \to R$$

There exists a set $S \subset \mathcal{P}$ of positive density such that for all $\ell \in S$

$$\begin{aligned} C \cap \ker\left[\mathrm{loc}_\ell:\ H^1(K,T) \to H^1(K_\ell,T)\right] &= \ker(\phi) \\ D \cap \ker\left[\mathrm{loc}_\ell:\ H^1(K,T^*) \to H^1(K_\ell,T^*)\right] &= \ker(\psi) \end{aligned}$$

Lemma 2.2.27. Assume R is a principal and satisfies Assumption (SI) and let $\mathcal{F}$ be a cartesian Selmer structure satisfying Assumptions (BI) and (NSD). Assume that

$$H^1_{\mathcal{F}}(K,T) \cong R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

for some $e_1 \geq \ldots \geq e_s \in \mathbb{N}$. Then there are infinitely many primes $\ell \in \mathcal{P}$ such that

$$H^1_{\mathcal{F}(\ell)}(K,T) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Proof. When $e_1 = \mathrm{length}(R)$, the result follows from Lemma 2.2.23. Hence we can assume without loss of generality that $e_1 < \mathrm{length}(R)$. Similarly, we can assume after Remark 2.2.24 that the core rank $\chi(\mathcal{F}) = 0$.

Similarly to the proof of Lemma 2.2.25, we can use Proposition 2.2.21 to find an auxiliary prime $q \in \mathcal{P}$ such that the localisation maps

$$\begin{aligned} \mathrm{loc}_q :\ & H^1_{\mathcal{F}}(K,T) \to H^1_{\mathrm{f}}(K_q,T)[\mathfrak{m}^{e_1}] \\ \mathrm{loc}_q :\ & H^1_{\mathcal{F}^*}(K,T^*) \to H^1_{\mathrm{f}}(K_q,T^*)[\mathfrak{m}^{e_1}] \end{aligned}$$

are surjective. Hence

$$H^1_{(\mathcal{F}^*)_q}(K,T^*) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Since $\chi(\mathcal{F}^q) = 1$ by Proposition 2.2.20, Proposition 2.2.16 implies that

$$H^1_{\mathcal{F}^q}(K,T) \cong R \times R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s} \tag{2.3}$$

By Proposition 2.2.26, we can find infinitely many primes ℓ satisfying the following:

- The kernel of the localisations loc_ℓ and loc_q defined on the dual Selmer group $H^1_{\mathcal{F}^*}(\mathbb{Q},T^*)$ are the same. Following Remark 2.1.13, we can define non-canonical isomorphisms

  $$H^1_{\mathrm{f}}(K_q,T^*) \cong H^1_{\mathrm{f}}(K_\ell,T^*) \cong T^*/(\tau-1)T^* \cong R \tag{2.4}$$

  Under this isomorphism, we can understand loc_q and loc_ℓ as elements in the dual space $H^1_{\mathcal{F}^*}(\mathbb{Q},T^*)^+$. In this setting, the above condition implies that there exists a unit $u \in R^\times$ such that $\mathrm{loc}_\ell = u\mathrm{loc}_q$.

- The localisation map

  $$\mathrm{loc}_\ell :\ H^1_{\mathcal{F}^q}(K,T) \to H^1_{\mathrm{f}}(K_\ell,T)$$

  is surjective.

  The first condition implies that

  $$H^1_{(\mathcal{F}^*)_{q\ell}}(K,T) = H^1_{(\mathcal{F}^*)_q}(K,T) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Since $\chi(\mathcal{F}^{q\ell}) = 2$ by Proposition 2.2.20, then Proposition 2.2.16 gives an isomorphism

$$H^1_{\mathcal{F}^{q\ell}}(K,T) \cong R^2 \times R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

By Proposition 2.2.6, we can consider the following exact sequence

$$H^1_{\mathcal{F}^{q\ell}}(K,T) \longrightarrow H^1_{\mathrm{s}}(K_q,T) \oplus H^1_{\mathrm{s}}(K_\ell,T) \longrightarrow H^1_{\mathcal{F}^*}(K,T^*)^\vee$$

Dualising the isomorphisms in (2.4), we obtain an isomorphism

$$H^1_{\mathrm{s}}(K_q,T) \oplus H^1_{\mathrm{s}}(K_\ell,T) \cong R^2 \tag{2.5}$$

such that the element $(1,-u^{-1})$ belongs to the kernel of the second map. Therefore, there is an element $z \in H^1_{\mathcal{F}^{\ell q}}(\mathbb{Q},T)$ such that $\mathrm{loc}_q(z) = 1$ and $\mathrm{loc}_\ell(z) = -u^{-1}$, under the identifications in (2.5)

It implies that the relaxed Selmer group splits as follows.

$$H^1_{\mathcal{F}^{q\ell}}(K,T) = Rz \oplus H^1_{\mathcal{F}^q}(K,T) = Rz \oplus H^1_{\mathcal{F}^\ell}(K,T) \tag{2.6}$$

We want to show now that

$$\Pi_\ell \circ \mathrm{loc}_\ell : \ H^1_{\mathcal{F}^\ell}(K,T) \to H^1_{\mathrm{f}}(K_\ell,T)$$

where $\Pi_\ell : \ H^1(K_\ell,T) \to H^1_{\mathrm{f}}(K_\ell,T)$ is the projection of Proposition 2.1.12, is also surjective.

Indeed, let $x \in H^1_{\mathcal{F}^\ell}(K,T)$ be such that $\pi^{k-1}x \neq 0$, where π is a generator of $\mathfrak{m}$. By (2.6), there is a unique decomposition $x = \alpha z + \beta$, where $\alpha \in R$ and $\beta \in H^1_{\mathcal{F}^q}(K,T)$. Since

$$H^1_{\mathcal{F}^\ell}(K,T) \cong R \times R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s}$$

then $\pi^{k-e_1}x \in H^1_{\mathcal{F}}(K,T) \subset H^1_{\mathcal{F}^q}(K,T)$ so $\alpha \in \mathfrak{m}^{e_1}$. Then $\pi^{k-1}\beta \neq 0$. By the assumptions on the prime ℓ,

$$\mathrm{loc}_\ell\big(H^1_{\mathcal{F}^q}(K,T)\big) = H^1_{\mathrm{f}}(K_\ell,T)$$

the isomorphism in (2.3) implies that $\mathrm{loc}_\ell(\beta)$ generates $H^1_{\mathrm{f}}(K_\ell,T)$. Indeed, every element of $H^1_{\mathcal{F}^q}(K,T)$ is a linear combination of β and elements of $\mathfrak{m}^{e_2}$-torsion, so loc_ℓ would only be surjective when $\mathrm{loc}_\ell(\beta)$ generates the whole $H^1_{\mathrm{f}}(K_\ell,T)$.

We have that

$$(\Pi_\ell \circ \mathrm{loc}_\ell)(x) - (\Pi_\ell \circ \mathrm{loc}_\ell)(\beta) = \alpha(\Pi_\ell \circ \mathrm{loc}_\ell)(z) \in \mathfrak{m}^{e_1} H^1_{\mathrm{f}}(K_\ell,T)$$

Then $(\Pi_\ell \circ \mathrm{loc}_\ell)(x)$ also generates $H^1_{\mathrm{f}}(K_\ell, T)$ and thus

$$\Pi_\ell \circ \mathrm{loc}_\ell : \ H^1_{\mathcal{F}^\ell}(K, T/\mathfrak{m}^k) \to H^1_{\mathrm{f}}(K_\ell, T)$$

is also surjective. Note that

$$H^1_{\mathcal{F}(\ell)}(K, T/\mathfrak{m}^k) = \ker\left(\Pi_\ell \circ \mathrm{loc}_\ell : H^1_{\mathcal{F}^\ell}(K, T/\mathfrak{m}^k) \to H^1_{\mathrm{f}}(K_\ell, T/\mathfrak{m}^k)\right)$$

Then the structure theorem over principal ideal domains implies that

$$H^1_{\mathcal{F}(\ell)}(K, T/\mathfrak{m}^k) \cong R/\mathfrak{m}^{e_2} \times \cdots \times R/\mathfrak{m}^{e_s} \qquad \square$$

2.3 Examples of Selmer modules

Simplest arithmetically interesting Selmer structures are defined when the coefficient ring R is either the p-adic numbers $\mathbb{Z}_p$ for some integer p or the ring of integers $\mathcal{O}$ of some p-adic field. We can then consider their propagations to the finite quotients T/p^k, as in Definition 2.2.7.

When the coefficient ring is a discrete valuation ring, we can extend Definition 2.1.9 to the ramified primes not dividing p.

Definition 2.3.1 (Finite cohomology)**.** Assume R is a discrete valuation ring of residue characteristic p. For every finite place of K not dividing p, the *finite* local condition is defined as

$$H^1_{\mathrm{f}}(K_\ell, T) := \ker\Big(H^1(K_\ell, T) \to H^1(\mathcal{I}_\ell, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\Big)$$

Proposition 2.3.2. ([Rub00, Lemma 1.3.5]) If the action of G_ℓ on T is unramified, the local conditions from Definitions 2.1.9 and 2.3.1 coincide for T and the propagation to a quotient of T.

Defining interesting local conditions at p is more complicated. The standard definition of finite cohomology, due to its similar behaviour, is the one introduced by S. Bloch and K. Kato in [BK90] using p-adic Hodge theory

Definition 2.3.3. Let $\ell \mid p$ be a place of K. The *finite* or *Bloch-Kato* local condition is defined as

$$H^1_{\mathrm{f}}(K_\ell, T) := \ker\Big(H^1(K_\ell, T) \to H^1(\mathcal{I}_\ell, T \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}})\Big)$$

where B_{crys} is Fontaine's period ring defined using p-adic Hodge theory.

That leads naturally to the definition of the Bloch-Kato Selmer structure

Definition 2.3.4. The Bloch-Kato Selmer structure is defined by the local conditions

$$\begin{cases} \mathbf{H}^1_{\mathcal{F}_{\mathrm{BK}}}(K_\ell, T) = \mathbf{H}^1_{\mathrm{f}}(K_\ell, T) & \text{if } \ell \nmid \infty \\ \mathbf{H}^1_{\mathcal{F}_{\mathrm{BK}}}(K_\ell, T) = \mathbf{H}^1(K_\ell, T) & \text{if } \ell \mid \infty \end{cases}$$

Remark 2.3.5. Since we are assuming that $p >$, the local cohomology groups at archimedean primes are always trivial.

Remark 2.3.6. The Bloch-Kato Selmer group is the one concernig the Bloch-Kato conjecture, saying that it is conjecturally related to the order of vanishing of the L-function (see Definition A.1.6) via the formula:

$$\mathrm{ord}_{s=1}\ L(T,s) = \mathrm{rank} H^1_{\mathcal{F}_{\mathrm{BK}}}(K, T^*)^\vee - \mathrm{rank}_{\mathbb{Z}_p}\ H^0(\mathbb{Q}, T^*)^\vee$$

Although the Bloch-Kato Selmer structure is arguably the most interesting one, there are other local conditions at primes dividing p that also play significant roles in this theory. The most basic example might be to consider the relaxed local condition at p, leading to the canonical Selmer structure.

Definition 2.3.7. The canonical Selmer structure is defined by the local conditions

$$\begin{cases} \mathbf{H}^1_{\mathcal{F}_{\mathrm{BK}}}(K_\ell, T) = \mathbf{H}^1_{\mathrm{f}}(K_\ell, T) & \text{if } \ell \nmid p\infty \\ \mathbf{H}^1_{\mathcal{F}_{\mathrm{BK}}}(K_\ell, T) = \mathbf{H}^1(K_\ell, T) & \text{if } \ell \mid p\infty \end{cases}$$

Chapter 3

Kolyvagin systems

This chapter is dedicated to outline the classical theory of Kolyvagin systems, developed in [MR04], that describes the Fitting ideals of Selmer groups of core rank 1 and and develop a theory that describes the Fitting ideals of Selmer groups defined from a Selmer structure $\mathcal{F}$ of core rank 0. The results in this chapter are a generalisation of my previous results [Ang25, Theorems 1.1 and 1.3].

The general strategy is to consider an auxiliary Selmer structure $\mathcal{G}$ of core rank one, which will admit a Kolyvagin system $\kappa \in \mathrm{KS}(\mathcal{G})$. We will use this Kolyvagin system to construct a sequence of ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$.

In the most general case, we can show (see Theorem 3.3.7) that the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$ coincide with the Fitting ideals of the Selmer group $\mathrm{Fitt}_{\mathcal{F}^*}(K, T^*)^\vee$ for at least half of the indices i, provided that κ is a primitive Kolyvagin system.

If we assume, in addition, that T is not a residually self-dual Galois representation, then it will be shown in Theorem 3.4.1 that the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$ coincide with the Fitting ideals of the Selmer group whenever κ is primitive.

3.1 Kolyvagin systems and theta ideals

In this section, we outline the classical theory of Kolyvagin systems, as described in [MR04]. This theory is limited to principal coefficient rings R and core rank being equal to one.

Assumption (CR1). Assume, in addition to Assumption (SI) that R is a principal ring, with π denoting a generator of the maximal ideal $\mathfrak{m}$. Moreover, consider a cartesian Selmer structure of core rank $\chi(\mathcal{F}) = 1$.

Notation 3.1.1. In order to simplify the notation, we fix a generator τ_ℓ of $\mathcal{G}_\ell = \mathrm{Gal}(K(\ell)/K)$ for every Kolyvagin prime $\ell \in \mathcal{P}$. This choice fixes, by Proposition 2.1.16, an isomorphism

$$\phi_\ell^{\mathrm{fs}} : H^1_{\mathrm{f}}(K_\ell, T) \to H^1_{\mathrm{s}}(K_\ell, T)$$

Definition 3.1.2. A *Kolyvagin system* for a Selmer structure $\mathcal{F}$

$$\kappa = \left\{\kappa_n \in H^1_{\mathcal{F}(n)}(K, T) : \ n \in \mathcal{N}\right\}$$

satisfying the following relation for every $n \in \mathcal{N}$ and $\ell \in \mathcal{P}$ not dividing n. By the definition of Selmer module, we have that

$$\begin{aligned} \mathrm{loc}_\ell(\kappa_n) &\in H^1_{\mathcal{F}(n)}(K_\ell, T) = H^1_{\mathrm{f}}(K_\ell, T) \\ \mathrm{loc}_\ell(\kappa_{n\ell}) &\in H^1_{\mathcal{F}(n\ell)}(K_\ell, T) = H^1_{\mathrm{tr}}(K_\ell, T) \end{aligned}$$

The collection κ is a Kolyvagin system if the following is satisfied

$$\mathrm{loc}_\ell(\kappa_{n\ell}) = \phi_\ell^{\mathrm{fs}} \circ \mathrm{loc}_\ell(\kappa_n) \tag{3.1}$$

for every $n \in \mathcal{N}$ and $\ell \in \mathcal{P}$ not dividing n.

Remark 3.1.3. The set of Kolyvagin systems has a natural structure of R-module. It will be denoted by $\mathrm{KS}(T, \mathcal{F})$, or simply $\mathrm{KS}(\mathcal{F})$ when there is no risk of confusion.

Kolyvagin systems carry information about the structure of the Selmer group. The key idea is to look at the classes κ_n, where $n \in \mathcal{N}_i$ for the different non-negative integers $i \in \mathbb{Z}_{\geq 0}$. The information carried by a single class κ_n is seen in its index.

Definition 3.1.4. Let M be an R-module and let $a \in M$. Consider the canonical map into the bidual module

$$\Phi : \ M \to M^{++} : a \in M \mapsto \left[\varphi \in \mathrm{Hom}(M, R) \mapsto \varphi(a)\right]$$

The *index* of a is defined as

$$\mathrm{ind}(a, M) := \mathrm{Im}(\Phi(a))$$

Notation 3.1.5. When there is no risk of confusion, we will denote $\mathrm{ind}(a)$ instead of $\mathrm{ind}(a, M)$.

When R is a simple ring, the index of an element has a more explicit description.

Remark 3.1.6. When R is a self-injective ring and $a \in M$, then

$$\operatorname{ind}(a, M) = R\big[R[a]\big]$$

Remark 3.1.7. When R is a discrete valuation ring and M is an R-module, we denote by M_{tors} the torsion submodule of M. If $a \in M$, then

$$\operatorname{ind}(a) = \mathfrak{m}^{\max\{j \in \mathbb{N}:\ a \in \mathfrak{m}^j M + M_{\text{tors}}\}}$$

We can now define the ideals Θ_i as the ideals of R generated by the indices of all κ_n where $n \in \mathcal{N}_i$.

Definition 3.1.8. Let $\kappa \in \mathrm{KS}(\mathcal{F})$. The theta ideals of κ are defined as

$$\Theta_i(\kappa) := \sum_{n \in \mathcal{N}_i} \operatorname{ind}\Big(\kappa_n, H^1_{\mathcal{F}(n)}(K, T)\Big)$$

3.2 Selmer structures of core rank 1

This section relates the theta ideals with the Fitting ideals of the Selmer group in the case where the core rank $\chi(\mathcal{F}) = 1$. This control comes from the fact that, when $\chi(\mathcal{F}) = 1$, the module of Kolyvagin systems is a free R-module of rank one.

Theorem 3.2.1. ([MR04, Theorem 4.3.3]) Suppose that Assumptions (BI) and (CR1) hold. Then the module of Kolyvagin systems $\mathrm{KS}(\mathcal{F})$ is a free, cyclic R-module.

This theorem guarantees the existence of Kolyvagin systems generating $\mathrm{KS}(\mathcal{F})$. We will see that the theta ideals of these generators carry information about the Selmer group.

Definition 3.2.2. A Kolyvagin system is said to be *primitive* if it generates $\mathrm{KS}(\mathcal{F})$ as an R-module.

We can now state the most important theorem about, which relates the theta ideals of a primitive Kolyvagin system with the (higher) Fitting ideals of the Selmer group.

Theorem 3.2.3. ([MR04, Theorem 4.5.9]) Let R be a principal, artinian, local ring with finite residue field, and let T be an $R[[G_K]]$-module, which is free and finitely generated as R-module and unramified only at finitely many places. Let $\mathcal{F}$ be a cartesian Selmer structure on T of core rank $\chi(\mathcal{F}) = 1$. If $\kappa \in \mathrm{KS}(\mathcal{F})$ is a primitive Kolyvagin system, then

$$\Theta_i(\kappa) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}^*}(K, T^*)) \ \forall i \in \mathbb{Z}_{\geq 0}$$

The proof of Theorem 3.2.3 is divided in the following two lemmas:

Lemma 3.2.4. ([MR04, Corollary 4.5.4]) Suppose that Assumption (CR1) holds and let $\kappa \in \mathrm{KS}(T)$ be a primitive Kolyvagin system. Then the following holds for every $n \in \mathcal{N}$:

$$\mathrm{ind}(\kappa_n) = \mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*(n)}(K, T^*)\big)$$

Lemma 3.2.5. Under Assumption (CR1), then

$$\mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}^*}(K, T^*)\big) = \sum_{n \in \mathcal{N}_i} \mathrm{Fitt}_R^0\big(H^1_{(\mathcal{F}^*)(n)}(K, T^*)\big)$$

Proof. We start by showing, for any $n \in \mathcal{N}_i$, the inclusion

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*(n)}(K, T^*)\big) \subset \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}^*}(K, T^*)\big)$$

Consider the exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^*_n}(K, T^*) \longrightarrow H^1_{\mathcal{F}^*}(K, T^*) \longrightarrow \textstyle\prod_{\ell | n} H^1_{\mathrm{f}}(K, T^*)$$

Since all the prime divisors of n are Kolyvagin primes, the last term is isomorphic to $R^{\nu(n)}$. The description of Fitting ideals over principal rings in Proposition B.2.6 (see also Remark B.2.8) implies that

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*_n}(K, T^*)\big) \subset \mathrm{Fitt}_R^{\nu(n)}\big(H^1_{\mathcal{F}^*}(K, T^*)\big)$$

Since $H^1_{\mathcal{F}^*_n}(K, T) \subset H^1_{\mathcal{F}^*(n)}(K, T)$, then

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*(n)}(K, T^*)\big) \subset \mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*_n}(K, T^*)\big)$$

Combining both inclusions, we conclude the proof of this inclusion, which leads to

$$\sum_{n \in \mathcal{N}_i} \mathrm{Fitt}_R^0\big(H^1_{(\mathcal{F}^*)(n)}(K, T^*)\big) \subset \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}^*}(K, T^*)\big)$$

In order to deal with the other inclusion, we need to construct, for each $i \in \mathbb{Z}_{\geq 0}$, a vertex $n_i \in \mathcal{N}_i$ such that

$$\mathrm{Fitt}^0_R\big(H^1_{(\mathcal{F}^*)(n_i)}(K, T^*)\big) = \mathrm{Fitt}^i_R\big(H^1_{\mathcal{F}^*}(K, T^*)\big)$$

Assume we we have a non-canonical isomorphism

$$H^1_{\mathcal{F}^*}(K, T^*) \cong R^r \times R/\mathfrak{m}^{e_1} \times \cdots R/\mathfrak{m}^{e_s}$$

An inductive application of Lemma 2.2.23 (see also Remark 2.2.24) construct vertices $n_i \in \mathcal{N}_i$ such that

$$H^1_{\mathcal{F}(n_i)}(K, T) \cong R^{r-i} \times R/\mathfrak{m}^{e_1} \times \cdots R/\mathfrak{m}^{e_s} \text{ for } i \leq r$$
$$H^1_{\mathcal{F}(n_{r+j})}(K, T) \cong R/\mathfrak{m}^{e_{j+1}} \times \cdots R/\mathfrak{m}^{e_s} \text{ for } j \geq 1$$

This proves the other inclusion, what concludes the proof of this lemma. $\square$

3.3 Selmer structures of rank 0

Throughout this section, we assume that R is a principal, artinian ring, satisfying Assumption (CR1).

When $\mathcal{F}$ is a cartesian Selmer structure of core rank 0, we cannot apply the argument above since the only Kolyvagin system is the trivial one.

Theorem 3.3.1. ([MR04, Theorem 4.2.2]) Let $\mathcal{F}$ be a cartesian Selmer structure such that $\chi(\mathcal{F}) = 0$. Then $\mathrm{KS}(\mathcal{F}) = 0$.

The method we will use for the computation of the Selmer module $H^1_{\mathcal{F}}(K, T)$ when $\chi(\mathcal{F}) = 0$ involves considering an auxiliary Selmer structure $\mathcal{G} \geq \mathcal{F}$, also cartesian, such that $\chi(\mathcal{G}) = 1$. One can show that $\mathcal{F}$ and $\mathcal{G}$ only differ in one local condition.

Notation 3.3.2. Let $\mathcal{F} \leq \mathcal{G}$ be Selmer structures. We say that $\mathcal{F} \leq \mathcal{G}$ if, for every prime $\ell \in \mathcal{P}_K$,

$$H^1_{\mathcal{F}}(K_\ell, T) \subset H^1_{\mathcal{G}}(K_\ell, T)$$

Proposition 3.3.3. Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. Then there exists a unique prime ℓ such that $H^1_{\mathcal{F}}(K_\ell, T) \subsetneq H^1_{\mathcal{G}}(K_\ell, T)$. Moreover, there is a non-canonical homomorphism

$$H^1_{\mathcal{G}/\mathcal{F}}(K_\ell, T) := H^1_{\mathcal{G}}(K_\ell, T) \big/ H^1_{\mathcal{F}}(K_\ell, T) \cong R$$

Proof. By Proposition 2.2.6, there is a global duality exact sequence for $\overline{T} := T \otimes k$

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{F}}\big(K,\overline{T}\big) \arrow[r] & H^1_{\mathcal{G}}\big(K,\overline{T}\big) \arrow[r] & \bigoplus_{q\in\Sigma_{\mathcal{F}}\cup\Sigma_{\mathcal{G}}} \dfrac{H^1_{\mathcal{G}}\big(K_q,\overline{T}\big)}{H^1_{\mathcal{F}}\big(K_q,\overline{T}\big)} \arrow[dll, out=-10, in=170] \\
& H^1_{\mathcal{G}^*}\big(K,\overline{T}^*\big)^\vee \arrow[r] & H^1_{\mathcal{F}^*}\big(K,\overline{T}^*\big)^\vee \arrow[r] & 0
\end{tikzcd}$$

Since $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$, Definition 2.2.14 and dimension counting imply that

$$\dim_k \left(\bigoplus_{q\in\Sigma_{\mathcal{F}}\cup\Sigma_{\mathcal{G}}} \frac{H^1_{\mathcal{G}}(K_q,\overline{T})}{H^1_{\mathcal{F}}(K_q,\overline{T})} \right) = 1$$

Therefore, there exists a unique prime ℓ such that $H^1_{\mathcal{F}}\big(K_\ell,\overline{T}\big) \subsetneq H^1_{\mathcal{G}}\big(K_\ell,\overline{T}\big)$. Hence $H^1_{\mathcal{F}}(K_\ell,T) \subsetneq H^1_{\mathcal{G}}(K_\ell,T)$.

For all other primes $q \neq \ell$, we can apply [MR04, Lemma 1.1.5], which says that for every pair of cartesian Selmer structures $\mathcal{F}$ and $\mathcal{G}$, the quantity

$$\text{length}\big(H^1_{\mathcal{G}}(K_q, T\otimes R/\mathfrak{m}^i)\big) - \text{length}\big(H^1_{\mathcal{F}}(K_q, T\otimes R/\mathfrak{m}^i)\big) \tag{3.2}$$

is linearly dependent on i. Since this quantity vanishes for $i = 1$, then

$$H^1_{\mathcal{F}}(K_q,T) = H^1_{\mathcal{G}}(K_q,T)\ \forall q \neq \ell$$

For the prime ℓ, [MR04, Lemma 1.1.5] implies that

$$\text{length}\big(H^1_{\mathcal{G}/\mathcal{F}}(K_q,T)\big) = \text{length}(R)$$

Consider the following composition, which coincides with the multiplication by π^{k-1}.

$$H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,T) \longrightarrow H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,\overline{T}) \xrightarrow{\pi^{k-1}} H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,T)$$

The first map is surjective by the propagation of Selmer structures and the second one is injective since $\mathcal{F}$ is cartesian. Hence, $H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,T)$ is an R-module of length k whose $\mathfrak{m}^{k-1}$-torsion induces a non-trivial quotient. Hence, the structure theorem implies that

$$H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,T) \cong R$$

□

Remark 3.3.4. If we choose a Kolyvagin prime, or any other prime ℓ such that $H^1_{/\mathcal{F}}(K_\ell, T) \cong R$, the Selmer structure $\mathcal{G} = \mathcal{F}^\ell$ is cartesian with $\chi(\mathcal{F}^\ell) = 1$, by Proposition 2.2.20.

Now, we describe a process in which Kolyvagin systems for $\mathcal{G}$ describe the Selmer module $H^1_{\mathcal{F}}(K, T)$. In order to do that, we need to localise the Kolyvagin systems at the prime at which $\mathcal{F}$ and $\mathcal{G}$ differ.

Definition 3.3.5. Let $\mathcal{F} \leq \mathcal{G}$ be two cartesian Selmer structures with $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$ differing at the prime ℓ and let $\kappa \in \mathrm{KS}(\mathcal{G})$. For every $n \in \mathcal{N}$, define the quantities δ associated with κ by

$$\delta_n(\kappa, \mathcal{F}) := \mathrm{loc}_\ell(\kappa_n) \in H^1_{\mathcal{G}/\mathcal{F}}(K_\ell, T) \cong R$$

The quantities δ_n can be used to define the Θ ideals of rank 0.

Definition 3.3.6. Let $\mathcal{F} \leq \mathcal{G}$ be two cartesian Selmer structures such that $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. Let ℓ be the prime at which they differ and let $\kappa \in \mathrm{KS}(\mathcal{G})$. We can define the ideals of R

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) := \sum_{n \in \mathcal{N}_i} \mathrm{ind}\Big(\delta_n(\kappa, \mathcal{F}), H^1_{\mathcal{G}/\mathcal{F}}(K_\ell, T)\Big)$$

The comparison between the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$ and the Fitting ideals of $H^1_{\mathcal{F}}(K, T)$ leads to the first main result of this thesis.

Theorem 3.3.7. Let R be a principal, artinian, local ring with finite residue field, let T be an $R[[G_K]]$-module, unramified only at finitely many places which is free and finitely generated as an R-module. Assume $\mathcal{F} \leq \mathcal{G}$ are cartesian Selmer structures on T satisfying that $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. If $\kappa \in \mathrm{KS}(\mathcal{G})$ is a Kolyvagin system, then

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}^i_R\big(H^1_{\mathcal{F}}(K, T)\big) \ \forall i \in \mathbb{Z}_{\geq 0} \tag{3.3}$$

Moreover, if κ is primitive and one of the following conditions is satisfied:x

(i) $i \leq \mathrm{rank}_R\big(H^1_{\mathcal{F}}(K, T)\big)$ (see Definition 3.3.8 below)

(ii) $\Theta_{i-1}^{(0)}(\kappa, \mathcal{F}) \subsetneq \mathrm{Fitt}^{i-1}_R\left(H^1_{\mathcal{F}}(K, T)\right)$

(iii) The Fitting ideal $\mathrm{Fitt}^{i-1}_R\big(H^1_{\mathcal{F}}(K, T)\big)$ does not vanish and there is a strict inclusion

$$\frac{\mathrm{Fitt}^{i-1}_R\big(H^1_{\mathcal{F}}(K, T)\big)}{\mathrm{Fitt}^{i}_R\big(H^1_{\mathcal{F}}(K, T)\big)} \subsetneq \frac{\mathrm{Fitt}^{i}_R\big(H^1_{\mathcal{F}}(K, T)\big)}{\mathrm{Fitt}^{i+1}_R\big(H^1_{\mathcal{F}}(K, T)\big)}$$

where the quotients are well-defined because all the Fitting ideals are non-zero and R is a principal artinian ring.

Then

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)\ \forall i \in \mathbb{Z}_{\geq 0}$$

It is important to recall what we understand as the rank of an R-module, which is not a standard notion when R is not an integral domain.

Definition 3.3.8. If M is an R-module, we define $\mathrm{rank}_R(M)$ as the maximal cardinality of an R-linearly independent set in M.

Remark 3.3.9. Note that the second condition in Theorem 3.3.7 implies that, for every pair of consecutives indices there is one of them, say i, such that

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)\ \forall i \in \mathbb{Z}_{\geq 0}$$

Similarly to the results in §3.1, the proof of Theorem 3.3.7 is divided in the following lemmas, which will be proven in the next subsections. Note that is is enough to prove it for primitive Kolyvagin systems.

Lemma 3.3.10. If $\kappa \in \mathrm{KS}(\mathcal{G})$ is a primitive Kolyvagin system and $n \in \mathcal{N}$, then

$$\mathrm{ind}(\delta_n) = \mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}(n)}(K,T)\big)$$

Lemma 3.3.11. If $\mathcal{F}$ is a cartesian

$$\sum_{n \in \mathcal{N}_i} \mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}(n)}(K,T)\big) \subset \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

Proof. Analogous to the similar inclusion in Lemma 3.2.5. □

In order to prove the other inclusion, whenever it holds, we will construct a vertex $n_i \in \mathcal{N}_i$ such that

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}(n_i)}(K,T)\big) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

Note that the equality holds trivially when $i < r$, since the Fitting ideal $\mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}^*}(K,T^*)\big)$ vanishes. For $i = r$, the equality is proven in the following lemma.

Lemma 3.3.12. There exists some vertex $n_r \in \mathcal{N}_r$ such that

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}(n_r)}(K,T)\big) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

Lemma 3.3.12 completes the proof, under assumption (i), of the equality

$$\Theta_i^{(0)}(\kappa,\mathcal{F}) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

Note that Lemma 3.3.12 determines the structure of the modified Selmer group. Indeed, assume there is a structural homomorphism

$$H^1_{\mathcal{F}}(K,T) \cong R^r \times R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

for some exponents $e_1 \geq \cdots \geq e_s$, all being at most $k-1$. Then there is a homomorphism

$$H^1_{\mathcal{F}(n_r)}(K,T) \cong R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

We can extend this construction to higher values of i.

Lemma 3.3.13. For every $j \geq 0$, we can either construct a vertex

- $n_{r+j} \in \mathcal{N}_{r+j}$ such that $H^1_{\mathcal{F}(n_{r+j})}(K,T) \cong R/\mathfrak{m}^{e_{j+1}} \times \cdots \times R/\mathfrak{m}^{e_s}$.
- $n_{r+j+1} \in \mathcal{N}_{r+j+1}$ such that $H^1_{\mathcal{F}(n_{r+j+1})}(K,T) \cong R/\mathfrak{m}^{e_{j+1}} \times \cdots \times R/\mathfrak{m}^{e_s}$.

In addition, if the index $i = r+j$ satisfies the third condition in Theorem 3.3.7 , the vertex n_{r+j} can be constructed.

Note that such vertices guarantee the equality in Theorem 3.3.7 for their respective indices.

Corollary 3.3.14. For the indices $k \geq 0$ such that there exists an element $n_r \in \mathcal{N}_r$ as in Lemma 3.3.13, there is an equality

$$\sum_{n \in \mathcal{N}_{r+k}} \mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}(n)}(K,T)\big) = \mathrm{Fitt}_R^{r+k}\big(H^1_{\mathcal{F}}(K,T)\big)$$

Corollary 3.3.14 proves the equality $\Theta_i^{(0)}(\kappa,\mathcal{F}) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$ under the first assumption. Indeed, if such equality does not hold for some index $i-1 = r+j-1$, the vertex n_{r+j-1} in Lemma 3.3.13 cannot be constructed. Hence, Lemma 3.3.13 guarantees the existence of the vertex n_{r+j}, which proves the equality for the index $i = r+j$ by Corollary 3.3.14.

This concludes the proof of Theorem 3.3.7. The inclusion is a direct consequence of Lemmas 3.3.10 and 3.3.11. Lemma 3.3.12 and Corollary 3.3.14 proves the equality if at least one of the assumptions holds.

Theorem 3.3.7 requires $\mathcal{G}$ to be a primitive Kolyvagin system. This fact can also be checked from the quantities δ_n.

Theorem 3.3.15. Let R be a principal, artinian, local ring with finite residue field, let T be an $R[[G_K]]$-module, unramified only at finitely many places which is free and finitely generated as an R-module. Assume $\mathcal{F} \leq \mathcal{G}$ are cartesian Selmer structures on T satisfying that $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. If $\kappa \in \mathrm{KS}(\mathcal{G})$, then κ is primitive if and only if there is $n \in \mathcal{N}$ such that

$$\operatorname{ind}\big(\delta_n(\kappa, \mathcal{F}), H^1_{\mathcal{G}/\mathcal{F}}(K, T)\big) = R$$

Proof. If κ is not primitive, there is a primitive Kolyvagin systems κ' such that $\kappa = a\kappa'$ and $a \in R \setminus R^\times$. Then

$$\operatorname{ind}\big(\delta_n(\kappa, \mathcal{F})\big) = (a)\operatorname{ind}\big(\delta_n(\kappa', \mathcal{F})\big) \subset aR \subsetneq R$$

Conversely, if κ is primitive, Lemma 3.3.10 reduces the proof to finding some $n \in \mathcal{N}$ such that

$$H^1_{\mathcal{F}(n)}(K, T) = 0$$

Since $(T \otimes k)^* = T[\mathfrak{m}]^*$, Propositions 2.2.13 and 2.2.16 reduce the problem to finding $n \in \mathbb{N}$ such that

$$H^1_{\mathcal{F}(n)}(K, T \otimes k) = 0$$

Such $n \in \mathcal{N}$ can be obtained by an inductive application of Lemma 2.2.23 on the k-representation $T \otimes k$. $\square$

3.3.1 Proof of Lemma 3.3.10

The proof of Lemma 3.3.10 involves comparing the indices of κ_n and δ_n, so we can then apply Lemma 3.2.4.

Lemma 3.3.16. For every $n \in \mathcal{N}$, let

$$C_n := \operatorname{coker}\big(\mathrm{loc}_\ell :\ H^1_{\mathcal{G}(n)}(K, T) \to H^1_{\mathcal{G}/\mathcal{F}}(K_\ell, T)\big)$$

Then $\operatorname{ind}(\delta_n) = \operatorname{ind}(\kappa_n) \cdot \mathrm{Fitt}^0_R(C_n)$.

Proof. Note that Propositions 2.2.16 and 2.2.20 imply the existence of a non-canonical isomorphism

$$H^1_{\mathcal{G}(n)}(K,T) \cong R \oplus H^1_{\mathcal{G}^*(n)}(K,T^*) \tag{3.4}$$

Let (x_n, y_n) be the components of κ_n under this identification. By Lemma 3.2.4, $\mathrm{ind}(\kappa_n) = \mathrm{Fitt}^0_R\big(H^1_{\mathcal{F}^*(n)}(K,T^*)\big)$. Since $\mathcal{F} \leq \mathcal{G}$, then

$$H^1_{\mathcal{G}^*(n)}(K,T^*) \subset H^1_{\mathcal{F}^*(n)}(K,T^*)$$

By Remark B.2.8 after Proposition B.2.6, that leads to

$$\mathrm{Fitt}^0_R\big(H^1_{\mathcal{F}^*(n)}(K,T^*)\big) \subset \mathrm{Fitt}^0_R\big(H^1_{\mathcal{G}^*(n)}(K,T^*)\big)$$

Hence $\mathrm{Ind}(\kappa_n)$ annihilates $H^1_{\mathcal{G}^*(n)}(K,T^*)$, so $y_n = 0$. In addition, x_n is a generator of $\mathrm{ind}(\kappa_n)$. The decomposition in (3.4) induces a map in R^+ defined by

$$R \rightarrowtail H^1_{\mathcal{G}(n)}(K,T) \xrightarrow{\mathrm{loc}_\ell} H^1_{\mathcal{G}/\mathcal{F}}(K_\ell,T) \xrightarrow{\sim} R$$

The composite map is the multiplication by some $a \in R$, which is also a generator of $\mathrm{Fitt}^0(C_n)$. Therefore,

$$\mathrm{ind}(\delta_n) = \mathrm{ind}(\kappa_n)\mathrm{Fitt}^0_R(C_n)$$

□

Proof of Lemma 3.3.10. The exact sequence in Proposition 2.2.6 induces a short exact sequence

$$0 \longrightarrow C_n \longrightarrow H^1_{\mathcal{F}^*(n)}(K,T)^\vee \longrightarrow H^1_{\mathcal{G}^*(n)}(K,T)^\vee \longrightarrow 0$$

By Corollary B.2.7, we have that

$$\mathrm{Fitt}^0_R\big(H^1_{\mathcal{F}^*(n)}(K,T)\big) = \mathrm{Fitt}^0_R\big(C_n\big)\mathrm{Fitt}^0_R\big(H^1_{\mathcal{G}^*(n)}(K,T)\big)$$

By Theorem 6.3.18 and Lemma 3.3.16, we obtain that

$$\mathrm{Fitt}^0_R\big(C_n\big)\,\mathrm{ind}(\kappa_n) = \mathrm{ind}(\delta_n)$$

where the second inequality follows from Lemma 3.2.4 and the last one from Lemma 3.3.16. □

3.3.2 Proof of Lemma 3.3.12

The proof is obtained as an inductive application of Lemma 2.2.23. By the structure theorem of finitely generated R-modules, and the definition of core rank, there are non-canonical homomorphisms

$$H^1_{\mathcal{F}}(K,T) = H^1_{\mathcal{F}^*}(K,T^*) = R^r \times R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

We will construct inductively a vertex $n_i \in \mathcal{N}_i$, where $i \leq r$, such that

$$H^1_{\mathcal{F}(n_i)}(K,T) \cong H^1_{\mathcal{F}^(n_i)}(K,T^*) \cong R^{r-i} \times R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Indeed, assume we have constructed $n_i \in \mathcal{N}_i$ for some i smaller than r. Clearly, $H^1_{\mathcal{F}(n_i)}(K,T)$ contains a submodule isomorphic to R, so Lemma 2.2.23 implies the existence of a prime ℓ_{i+1} such that for $n_{i+1} = n_i\ell_{i+1}$, we get that

$$H^1_{\mathcal{F}^(n_{i+1})}(K,T^*) \cong R^{r-(i+1)} \times R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Since $\chi(\mathcal{F}) = 0$, there is a non-canonical homomorphism

$$H^1_{\mathcal{F}(n_{i+1})}(K,T) \cong H^1_{\mathcal{F}^*(n_{i+1})}(K,T^*) \cong R^{r-(i+1)} \times R/\mathfrak{m}^{e_1} \times \cdots \times R/\mathfrak{m}^{e_s}$$

3.3.3 Proof of Lemma 3.3.13

Note that the element $n_r \in \mathcal{N}_r$ was already constructed in Lemma 3.3.12. An inductive application of Lemma 2.2.25, implies the existence of an element $n_{r+j} \in \mathcal{N}_{r+j}$ such that

$$H^1_{\mathcal{F}(n_{r+j})}(K,T) \cong R/\mathfrak{m}^{t_{j+1}} \times R/\mathfrak{m}^{e_{j+2}} \times \cdots \times R/\mathfrak{m}^{e_s}$$

where $t_{j+1} \geq e_{j+1}$. We assume that we construct this elements minimising the exponents t_{j+1} in every step.

For an index j, if $t_{j+1} = e_{j+1}$, then the element $n_{r+j} \in \mathcal{N}_{r+j}$ satisfy the hypothesis of Lemma 3.3.13.

Otherwise, if $t_{j+1} > e_{j+1}$, then $t_{j+1} > e_{j+2}$ as well , so the last remark in Lemma 2.2.25 guarantees the existence of a prime ℓ_{r+j+1} such that, when $n_{r+j+1} = n_{r+j}\ell_{r+j+1}$, the exponent t_{j+2} coincides with e_{j+2}. By the minimality assumption on t_{j+2}, we know that

$$H^1_{\mathcal{F}(n_{r+j+1})}(K,T) \cong R/\mathfrak{m}^{e_{j+2}} \times \cdots \times R/\mathfrak{m}^{e_s}$$

Finally, note that if $\mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}}(K,T)\big) \neq 0$, then $i \geq r$, so there is some $j \geq 0$ such that $i = r + j$. The condition

$$\frac{\mathrm{Fitt}_R^{r+j}\big(H^1_{\mathcal{F}}(K,T)\big)}{\mathrm{Fitt}_R^{r+j+1}\big(H^1_{\mathcal{F}}(K,T)\big)} \subsetneq \frac{\mathrm{Fitt}_R^{r+j+1}\big(H^1_{\mathcal{F}}(K,T)\big)}{\mathrm{Fitt}_R^{r+j+2}\big(H^1_{\mathcal{F}}(K,T)\big)}$$

can be seen as $e_{j+1} > e_{j+2}$. Then, we are in the $t_{j+1} > e_{j+2}$ case, so the vertex n_{r+j+1} can be constructed.

3.4 Non-self-dual representations

This section is devoted to the computation of the Fitting ideals of a Selmer module $H^1_{\mathcal{F}}(K,T)$ of a cartesian Selmer structure of core rank $\chi(\mathcal{F}) = 0$, defined for a Galois representation T with is not residually self-dual. More specifically, we assume that T satisfies Assumptions (NSD).

Theorem 3.4.1. Let R be a principal, artinian, local ring with finite residue field, let T be an $R[[G_K]]$-module unramified only at finitely many places and satisfying Assumptions (NSD), and let $\mathcal{F} \leq \mathcal{G}$ be a cartesian Selmer structures on T such that $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$. If $\kappa \in \mathrm{KS}(\mathcal{G})$ is a primitive Kolyvagin system, then

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i(H^1_{\mathcal{F}}(K,T))$$

Proof. Note that the inclusion

$$\Theta_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}_R^i(H^1_{\mathcal{F}}(K,T))$$

is proven by Theorem 3.3.7. In addition, Lemma 3.3.10 reduces the proof of the other inclusion to the following lemma. □

Lemma 3.4.2. For every $i \in \mathbb{Z}_{\geq 0}$, there exists some vertex $n_i \in \mathcal{N}_i$ such that

$$\mathrm{Fitt}_R^0\big(H^1_{\mathcal{F}^*(n_i)}(K,T)\big) = \mathrm{Fitt}_R^i\big(H^1_{\mathcal{F}}(K,T)\big)$$

Proof. Assume we we have a non-canonical isomorphism

$$H^1_{\mathcal{F}}(K,T^*) \cong R^r \times R/\mathfrak{m}^{e_1} \times \cdots R/\mathfrak{m}^{e_s}$$

An inductive application of Lemma 2.2.27 (see also Remark 2.2.24) construct vertices $n_i \in \mathcal{N}_i$ such that

$$\begin{aligned} H^1_{\mathcal{F}(n_i)}(K,T) &\cong R^{r-i} \times R/\mathfrak{m}^{e_1} \times \cdots R/\mathfrak{m}^{e_s} \text{ for } i \leq r \\ H^1_{\mathcal{F}(n_{r+j})}(K,T) &\cong R/\mathfrak{m}^{e_{j+1}} \times \cdots R/\mathfrak{m}^{e_s} \text{ for } j \geq 1 \end{aligned}$$

The proof of this lemma is thus concluded by Proposition B.2.6 (see also Remark B.2.8). □

Part II

Ultra Kolyvagin systems

Chapter 4

Patched Cohomology

In this chapter, we present the construction of patched Galois cohomology, which is based on the set-theoretic notion of ultrafilters. This construction was originally introduced in [Swe22]. There are no major new results included in this chapter, since the construction and properties of local and global patched Galois cohomology were already introduced in loc. cit. However, we develop here the global duality exact sequence of patched Selmer groups (see Proposition 4.3.11), which will be required to develop the theory of ultra Kolyvagin systems.

4.1 Ultrafilters

In this section, we develop the basic theory of ultrafilters and their use to construct ultraproducts, which will be the basic tool to construct the patched cohomology in the subsequent sections.

4.1.1 Definition

Definition 4.1.1. A *filter* in the natural numbers is a collection of sets $\mathcal{U}$ in the power set $\mathcal{P}(\mathbb{N})$ such that

- (F0) The empty set does not belong to $\mathcal{U}$.
- (F1) If $S_1 \subset S_2$ and $S_1 \in \mathcal{U}$, then $S_2 \in \mathcal{U}$.
- (F2) If $S_1, S_2 \in \mathcal{U}$, then $S_1 \cap S_2 \in \mathcal{U}$.

We say that a filter is an *ultrafilter* if it also satisfies the following condition

- (UF) For every set $S \in \mathcal{P}(\mathbb{N})$, either $S \in \mathcal{U}$ or $\mathbb{N} \setminus S \in \mathcal{U}$.

The key property of ultrafilters is that (UF) generalises to finite partitions, i.e., ultrafilters contain exactly one set in every finite partition of $\mathbb{N}$.

Proposition 4.1.2. Let $\mathcal{U}$ be an ultrafilter and let $\{P_1, \ldots, P_s\}$ be a partition of $\mathbb{N}$. Then there exists a unique i such that $P_i \in \mathcal{U}$.

Proof. It follows from an inductive application of (UF). $\square$

Last proposition can be reinterpreted in the following form:

Corollary 4.1.3. ([Swe22, Proposition 2.1.2]) Let $\mathcal{U}$ be an ultrafilter, let $S \in \mathcal{U}$ and let C be a finite set. For every map $f : S \to C$, there exists a unique $c \in C$ such that $f^{-1}(c) \in \mathcal{U}$.

Notation 4.1.4. When we say that some property is satisfied for $\mathcal{U}$-many $i \in \mathbb{N}$, we mean that there is a set $S \in \mathcal{U}$ such that the property holds for every $i \in \mathcal{U}$.

The only ultrafilters we can explicitly describe are those formed by the subsets of the naturals containing one specific element. They are known as principal ultrafilters. However, they are the least interesting ones for our purposes.

Definition 4.1.5. Let $a \in \mathbb{N}$. The collection of sets

$$\mathcal{U}_a = \{S \subset \mathbb{N} : \; a \in S\}$$

is an ultrafilter. These are known as *principal ultrafilters*.

In fact, principal ultrafilters are the only ones containing finite sets.

Proposition 4.1.6. Let $\mathcal{U}$ be an ultrafilter and assume there is a finite set S that belongs to $\mathcal{U}$. Then there exists an element $a \in S$ such that $\mathcal{U} = \mathcal{U}_a$.

Proof. Consider the finite union

$$\mathbb{N} = (\mathbb{N} \setminus S) \cup \bigcup_{a \in S} \{a\}$$

By Proposition 4.1.2, one of the above sets belongs to $\mathcal{U}$. Since $(\mathbb{N} \setminus S)$ does not, there is some $a \in S$ such that $\{a\} \in \mathcal{U}$. By (F1), $\mathcal{U}_a \subset \mathcal{U}$.

In order to show the equality, assume there exists $T \in \mathcal{U} \setminus \mathcal{U}_a$. By (F2), $T \cap \{a\} = \emptyset \in \mathcal{U}$, contradicting (F0). Therefore, $\mathcal{U} = \mathcal{U}_a$. $\square$

As mentioned above, those ultrafilters which are interesting for our purposes are the non-principal ones. Although they cannot be explicitly constructed, its existence is guaranteed, assuming the axiom of choice, by the analogy between ultrafilters and maximal ideals shown in Proposition 4.1.11 below. They are those ultrafilters containing the Fréchet filter consisting of sets with finite complement.

Definition 4.1.7. The *Fréchet filter* is the collection of subsets of the natural numbers defined as

$$\mathcal{F} = \{S \subset \mathbb{N} : \ \mathbb{N} \setminus S \text{ finite}\}$$

4.1.2 Analogy between ultrafilters and ideals

The set $\mathcal{P}(\mathbb{N})$ can be endowed with a natural structure of a boolean ring. In order to do that, we define a set-theoretic bijection to the functions on the naturals with values on the finite field with two elements $\mathbb{F}_2$:

$$\mathbb{P}(\mathbb{N}) \to \mathcal{C}(\mathbb{N}, \mathbb{F}_2) : \ A \mapsto 1 - \chi_A$$

where χ_A is the characteristic function. The natural boolean structure in $\mathcal{C}(\mathbb{N}, \mathbb{F}_2)$ induces, via the above bijection, a boolean ring structure in $\mathcal{P}(\mathbb{N})$. It is possible to explicitly describe the operations in $\mathbb{P}(\mathbb{N})$.

Definition 4.1.8. The *filtered* boolean structure $\mathcal{B}(\mathbb{N})$ in $\mathbb{P}(\mathbb{N})$ is given by the operations $+$ and $\cdot$ defined by

$$A + B = A \Delta B = (A \cap B) \cup (A^c \cap B^c), \quad A \cdot B = A \cup B$$

where S^c denotes the complement set and Δ denotes the symmetric difference.

Remark 4.1.9. The boolean ring structure in 4.1.8 is not the standard one appearing in the literature, but it can be recovered by conjugating via the involution obtained by sending each set to its complement.

We can now identify filters and ultrafilters with ideals and maximal ideals[1].

Proposition 4.1.10. The filters coincide with the ideals strictly contained in the boolean ring $\mathcal{B}(\mathbb{N})$.

[1]With the standard convention, filters (resp. ultrafilters) are the set of complements of ideals (resp. maximal ideals)

Proof. Let $\mathcal{F}$ be a filter in $\mathbb{P}(\mathbb{N})$ and let $A, B \in \mathcal{F}$. Then

$$A + B = (A \cap B) \cup (A^c \cap B^c) \supset A \cap B \in \mathcal{F}$$

by (F2). Therefore, $A + B \in \mathcal{F}$ by (F1). If $T \subset \mathbb{N}$, then

$$A \cdot T = A \cup T \supset A \in \mathcal{F}$$

by (F1). Finally, (F0) implies that $\mathcal{F}$ is not the full $\mathcal{P}(\mathbb{N})$.

Conversely, assume $\mathcal{F}$ is an ideal strictly contained in $\mathbb{P}(\mathbb{N})$. Since the unit element in $\mathcal{B}(\mathbb{N})$ is the empty set, then (F0) needs to hold. Assume that $S \in \mathcal{F}$ and $S \subset T$, then $T = S \cdot T$, so it belongs to $\mathcal{F}$, which proves (F1). Finally, if $A, B \in \mathcal{F}$, then

$$A \cap B = (A \Delta B) \Delta (A \cup B) = A + B + AB \in \mathcal{F}$$

Then (F2) holds. $\square$

Proposition 4.1.11. The ultrafilters in $\mathcal{P}(\mathbb{N})$ are exactly the maximal ideals of $\mathcal{B}(\mathbb{N})$.

Proof. Let $\mathcal{U}$ be an ultrafilter. By Proposition 4.1.10, $\mathcal{U}$ is an ideal of $\mathcal{B}(\mathbb{N})$. Let $A = \mathbb{P}(\mathbb{N})/\mathcal{U}$ be its quotient ring. Note that, for any $S \subset \mathbb{N}$, then either $S \in \mathcal{U}$ or

$$S = \emptyset + S^c \in \emptyset + \mathcal{U}$$

because $S^c \in \mathcal{U}$ by (UF). Hence A is the ring with two elements, so it is a field and hence $\mathcal{U}$ is a maximal ideal.

Conversely, assume $\mathcal{U}$ is a maximal ideal, so $A = \mathbb{P}(\mathbb{N})/\mathcal{U}$ is a boolean field. Then $A = \mathbb{F}_2$ since every element is a root of $x(x-1)$. Then, for any $S \subset \mathbb{N}$, either $S \in \mathcal{U}$ or $1 + S \in \mathcal{U}$. This is equivalent to (UF) since $1 + S = \emptyset + S = S^c$. $\square$

4.1.3 Ultraproducts

In this section, we will use the concept of ultrafilters to patch sequences of sets. This patching can be endowed with algebraic structures in case the original sets also had those structures.

Notation 4.1.12. Denote by $\mathcal{C}$ a category whose objects are sets and whose morphisms are functions. We denote by $\mathcal{C}^{\mathbb{N}}$ the category whose objects are sequences $(A_n)_{n\in\mathbb{N}}$ of objects in $\mathcal{C}$ and the morphisms from $(A_n)_{n\in\mathbb{N}}$ to $(B_n)_{n\in\mathbb{N}}$ are sequences of morphisms $A_n \to B_n$.

Definition 4.1.13. Let $\mathcal{U}$ be an ultrafilter and let $(M_n)_{n\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$. The *ultraproduct* $\mathcal{U}_n(M_n)$ is the set defined as

$$\mathcal{U}_n(M_n) = \prod_{n\in\mathbb{N}} M_n \Big/\sim$$

where the product is the direct product of sets and $\sim$ is the equivalence relation defined as $(m_n) \sim (m'_n)$ if $m_n = m'_n$ for $\mathcal{U}$-many n.

Notation 4.1.14. We will also denote the ultraproduct by $\mathcal{U}(M_n)$ when there is no risk of confusion.

The ultraproduct has functorial properties.

Proposition 4.1.15. (Functoriality of the ultraproduct, [Swe22, Proposition 2.1.4]) The ultraproduct $\mathcal{U}$ defines a functor $\mathcal{C}^{\mathbb{N}} \to \mathbf{Set}$.

Proof. Let $\varphi :\ (A_n) \to (B_n)$ be a morphism in $\mathcal{C}^{\mathbb{N}}$. By definition, φ is a collection of morphisms $\varphi_n :\ A_n \to B_n$. Their product induces a morphism

$$\overline{\varphi} = \prod_{n\in\mathbb{N}} \varphi_n :\ \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} B_n$$

This product morphism restricts well to the ultraproduct, resulting in a map

$$\varphi^{\mathcal{U}} :\ \mathcal{U}(A_n) \to \mathcal{U}(B_n)$$

Indeed, if $(\alpha_i), (\alpha'_i) \in \prod_{n\in\mathbb{N}} A_n$ are two equivalent sequences, then $\alpha_i = \alpha'_i$ for $\mathcal{U}$-many i. Then $\varphi(\alpha_i) = \varphi(\alpha'_i)$ for $\mathcal{U}$-many i. It implies that $\overline{\varphi}(\alpha_i)$ and $\overline{\varphi}(\alpha'_i)$ are equivalent sequences in $\prod_{n\in\mathbb{N}} B_n$. Hence, $\varphi^{\mathcal{U}}$ is well defined and, since it clearly behaves well with the composition, the ultraproduct is functorial. $\square$

The ultraproduct has only been defined as a set. However, sometimes this approach loses information compared to consider the ultraproduct as an object in the category $\mathcal{C}$.

Definition 4.1.16. An *ultra category* $\mathcal{C}$ is a category whose objects are sets, whose morphisms are functions, and endowed with a functor

$$\mathcal{U} :\ \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$$

such that the following diagram is commutative:

$$
\begin{tikzcd}
\mathcal{C}^{\mathbb{N}} \arrow[r, "\mathcal{U}"] \arrow[dr] & \mathcal{C} \arrow[d] \\
& \mathbf{Set}
\end{tikzcd}
$$

where the functor $\mathcal{C}^{\mathbb{N}} \to \mathbf{Set}$ is the ultraproduct defined in Definition 4.1.13 and $\mathcal{C} \to \mathbf{Set}$ is the forgetful sending every object to its underlying set.

Remark 4.1.17. Pointed sets, groups, rings, modules are examples of ultra categories, where the functor $\mathcal{U} : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ is obtained by endowing the ultraproduct in Definition 4.1.13 with the natural operations.

Remark 4.1.18. As we will show in Corollary 4.1.22, the category of finite sets bounded by some fixed constant is an ultra category. That is arguably the most useful technical property of the ultraproduct.

Notation 4.1.19. If M is a set, we will denote by $\mathcal{U}(M)$ the ultraproduct of the sequence (M_n) in which $M_n = M$ for all n.

Proposition 4.1.20. ([Swe22, proposition 2.1.5]) Assume $\mathcal{C}$ is a category of pointed sets. Then the ultraproduct $\mathcal{U}$ is an exact functor.

Proof. We will denote by 0 the distinguished point in every object of $\mathcal{C}$. Assume we have an exact sequence in $\mathcal{C}^{\mathbb{N}}$,

$$0 \longrightarrow (A_i) \xrightarrow{\overline{\mu}} (B_i) \xrightarrow{\overline{\varepsilon}} (C_i) \xrightarrow{0} 0$$

We start by showing the injectivity of $\mu^{\mathcal{U}}$. Let $\alpha = (\alpha_i) \in \ker(\mu^{\mathcal{U}})$, which means that $\mu_i(\alpha_i) = 0$ for $\mathcal{U}$-many i. Since μ_i are injective maps, then $\alpha_i = 0$ for $\mathcal{U}$-many i, so $(\alpha_i) \sim 0$ in $\mathcal{U}(A_n)$. Thus $\mu^{\mathcal{U}}$ is injective.

The composition of $\varepsilon^{\mathcal{U}} \circ \mu^{\mathcal{U}}$ vanishes since $\overline{\varepsilon} \circ \overline{\mu}$ also does. Conversely, let $\beta = (\beta_n) \in \ker(\varepsilon^{\mathcal{U}})$. Let $S_\beta \in \mathcal{U}$ be the set of indices such that $\varepsilon_i(\beta_i) = 0$. For those indices, $\beta_i \in \mathrm{Im}(\mu_i)$. We can thus define $\alpha = (\alpha_i)$ by

$$\begin{cases} \alpha_i \in \mu_i^{-1}(\beta_i) & \text{if } i \in S_\beta \\ \alpha_i = 0 & \text{if } i \notin S_\beta \end{cases}$$

Clearly $(\mu_i(\alpha_i))$ is equivalent to (β_i), so $\beta \in \mu^{\mathcal{U}}$.

Finally, the surjectivity of $\varepsilon^{\mathcal{U}}$ follows from being a quotient map of the surjective map $\overline{\varepsilon}$. $\square$

In general, it is difficult to compute the ultraproduct, but there is a special case in which it is explicit: the ultraproduct of a sequence of finite sets of bounded order.

Lemma 4.1.21. Let (M_n) be a sequence of sets satisfying that there is a finite set such that $M_n = M$ for all n. Then the diagonal map $\Delta : M \to \mathcal{U}(M_n)$ is an isomorphism.

Proof. By (F0), the above map is clearly injective, so we only need to check surjectivity. A sequence $(m_n) \in \prod_{n\in\mathbb{N}} M$ induces a map

$$f : \mathbb{N} \to M : n \mapsto m_n$$

Since M is finite, Corollary 4.1.3 implies that there exists a unique $m \in M$ such that $f^{-1}(\{m\}) \in \mathcal{U}$. Hence (m_n) is equivalent to the constant sequence (m), so it belongs to the image of Δ. $\square$

Corollary 4.1.22. Let M_n be a sequence of finite sets whose orders are bounded above by some constant C. Then the ultraproduct $\mathcal{U}(M_n)$ is finite with order bounded by C.

Proof. Let S be a set of cardinality C. For each $n \in \mathbb{N}$, fix an injection

$$\mu_n : M_n \hookrightarrow S$$

By Proposition 4.1.20 and Lemma 4.1.21, there is an injection

$$\mathcal{U}(M_n) \hookrightarrow \mathcal{U}(S) \cong S$$

Thus, $\mathcal{U}(M_n)$ is finite with order bounded by C. $\square$

The ultraproduct also behaves well with direct products.

Proposition 4.1.23. If (A_n) and (B_n) are sequences of sets, modules or rings, there is a canonical identification

$$\mathcal{U}(A_n \times B_n) = \mathcal{U}(A_n) \times \mathcal{U}(B_n)$$

Proof. The canonical map

$$\prod_{n\in\mathbb{N}} A_n \times B_n \to \prod_{n\in\mathbb{N}} A_n \times \prod_{n\in\mathbb{N}} B_n,\ (a_n, b_n)_{n\in\mathbb{N}} \mapsto \left[(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right]$$

respects the equivalence relation given by the ultrafilter. Indeed, if two sequences $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ and $\{(a'_n, b'_n)\}_{n\in\mathbb{N}}$ are equivalent, the set

$$S := \{n \in \mathbb{N} : \ (a_n, b_n) = (a'_n, b'_n)\} \in \mathcal{U}$$

Therefore, by (F1),

$$S_a = \{n \in \mathbb{N} : \ a_n = a'_n\} \in \mathcal{U}$$

since it contains S. Hence (a_n) is equivalent to (a'_n). Similarly, (b_n) and (b'_n) are also equivalent. Then the map $\mathcal{U}(A_n \times B_n) \to \mathcal{U}(A_n) \times \mathcal{U}(B_n)$ is well defined.

Conversely, if we have two pairs of equivalent sequences (a_n) and (a'_n), and (b_n) and (b'_n). We denote by S_a and S_b the set of indices in which they coincide, then

$$\{n \in \mathbb{N} : (a_n, b_n) = (a'_n, b'_n)\} = S_a \cap S_b \in \mathcal{U}$$

by (F2). Then (a_n, b_n) and (a'_n, b'_n) are two equivalent sequences in $(A_n \times B_n)$, so the inverse map $\mathcal{U}(A_n) \times \mathcal{U}(B_n) \to \mathcal{U}(A_n \times B_n)$ is also well defined. □

The ultraproduct does not commute with the group of homomorphisms in general, but there exists an injective comparison map.

Proposition 4.1.24. Let (A_n), (B_n) be two sequences in $\mathcal{C}^{\mathbb{N}}$. Then there is an injection

$$\Psi : \ \mathcal{U}(\mathrm{Hom}(A_n, B_n)) \hookrightarrow \mathrm{Hom}\big(\mathcal{U}(A_n), \mathcal{U}(B_n)\big)$$

Proof. Let (φ_n) be the sequence representing an element in the ultraproduct $\mathcal{U}(\mathrm{Hom}(A_n, B_n))$ and let (a_n) be a sequence representing an element in $\mathcal{U}(A_n)$. We assign the sequence $\varphi_n(a_n) \in \mathcal{U}(B_n)$. Clearly, this assignment behaves well under the ultrafilter equivalences in both $\mathcal{U}(\mathrm{Hom}(A_n, B_n))$ and $\mathcal{U}(A_n)$, so it defines a map

$$\Psi : \mathcal{U}(\mathrm{Hom}(A_n, B_n)) \to \mathrm{Hom}\big(\mathcal{U}(A_n), \mathcal{U}(B_n)\big)$$

In order to check injectivity, consider another sequence (ψ_n) induces the same element in $\mathrm{Hom}\big(\mathcal{U}(A_n), \mathcal{U}(B_n)\big)$. For the sake of contradiction, assume that (φ_n) and (ψ_n) are not equivalent, i.e., the set

$$S = \{n \in \mathbb{N} : \ \varphi_n \neq \psi_n\} \in \mathcal{U}$$

Choose a sequence $(a_n) \in (A_n)$ where $\varphi_n(a_n) \neq \psi_n(a_n)$ for every $n \in S$. Hence $(\varphi_n(a_n))$ and $(\psi_n(a_n))$ represent different elements in $\mathcal{U}(B_n)$, so $\Psi(\varphi_n) \neq \Psi(\psi_n)$. □

The above map is an isomorphism when (B_n) is a constant sequence of finite sets.

Corollary 4.1.25. Let (A_n) be a sequence in $\mathcal{C}^{\mathbb{N}}$ and let $B \in \mathcal{C}$ which is also a finite set. Then there is an isomorphism

$$\Psi: \; \mathcal{U}(\mathrm{Hom}(A_n, B)) \to \mathrm{Hom}\big(\mathcal{U}(A_n), B\big)$$

Proof. By Lemma 4.1.21 and Proposition 4.1.24, the map Ψ is well-defined and injective. Since every element in $\mathrm{Hom}\big(\mathcal{U}(A_n), B\big)$ lifts to a homomorphism in

$$\mathrm{Hom}\left(\prod_{n\in\mathbb{N}} A_n, B\right) = \prod_{n\in\mathbb{N}} (\mathrm{Hom}(A_n, B))$$

then it is also in the image of Ψ. $\square$

4.1.4 Ultraprimes

An example of an ultraproduct of infinite sets leads to the concept of ultraprimes, which are the elements in the ultraproduct of the constant sequence of the set of prime numbers. We will not attempt to give a description of this ultraproduct, but its elements will play an important role in this theory.

Fix a non-principal ultrafilter $\mathcal{U}$ and a number field K. Denote by $\mathbb{P}_K$ the set of primes in K.

Definition 4.1.26. An *ultraprime* $\mathfrak{u}$ is an element of $\mathcal{U}(\mathbb{P}_K)$. More specifically, it is represented by a sequence of prime ideals $(\ell_n)_{n\in\mathbb{N}}$, and two sequences represent the same ultraprime if they coincide in $\mathcal{U}$-many primes.

Remark 4.1.27. The primes $\mathbb{P}_K$ are contained in the ultraprimes $\mathcal{U}(\mathbb{P}_K)$ via the diagonal map, i.e., a prime ℓ is identified with the equivalence class of the constant sequence (ℓ). The elements in the image of the map $\mathbb{P}_K \hookrightarrow \mathcal{U}(\mathbb{P}_K)$ are sometimes referred as *constant ultraprimes.*

We can use Corollary 4.1.3 to define the Frobenius element associated with an ultraprime $\mathfrak{u}$ in the absolute Galois group G_K.

Proposition 4.1.28. Let $\mathfrak{u} = (\ell_n)$ be an ultraprime and let L/K be a finite Galois extension of number fields. Then there exists a unique conjugacy class in $\mathrm{Gal}(L/K)$ such that $\mathrm{Frob}_{\ell_n}|_{L/K} \in c$ for

$\mathcal{U}$-many n. This class is called the *Frobenius conjugacy class* of $\mathfrak{u}$ at L/K and any element in it is called a *Frobenius automorphism.*

Proof. The sequence (ℓ_n) defines a map

$$F:\ \mathbb{N} \to \mathrm{Gal}(L/K)/\sim:\ n \mapsto [\mathrm{Frob}_{\ell_n}]$$

where $[\sigma]$ denotes the conjugacy class containing σ. Since $\mathrm{Gal}(L/K)$ is finite, Corollary 4.1.3 says that there exists a unique conjugacy class c such that $F^{-1}(\{c\}) \in \mathcal{U}$, which is defined to be the Frobenius conjugacy class $\mathrm{Frob}_{\mathfrak{u}}|_{L/K}$.

If we take an equivalent sequence ℓ'_n, the set

$$S = \{n \in \mathbb{N}:\ \ell_n = \ell'_n\} \in \mathcal{U}$$

Then, by (F2) and (F1)

$$S \cap F^{-1}(\{\sigma\}) \subset \{n \in \mathbb{N}:\ \mathrm{Frob}_{\ell'_n} = \sigma\} \in \mathcal{U}$$

Hence the definition of $\mathrm{Frob}_{\mathfrak{u}}\ |_{L/K}$ is independent of the sequence representing it. □

Definition 4.1.29. Let $\mathfrak{u} = (\ell_n)$ be an ultraprime. The (absolute) Frobenius conjugacy class $\mathrm{Frob}_{\mathfrak{u}}$ is defined as the conjugacy class

$$\mathrm{Frob}_{\mathfrak{u}} = \big(\mathrm{Frob}_{\mathfrak{u}}|_{L/K}\big)_{L/K} \subset \varprojlim_{L/K} \mathrm{Gal}(L/K) = G_K$$

Any automorphism in this class is called a Frobenius element $\mathrm{Frob}_{\mathfrak{u}}$.

Remark 4.1.30. In order to guarantee that Definition 4.1.29 is consistent, we need to show that $\mathrm{Frob}_{\mathfrak{u}}$ behaves well under the restriction of finite extensions L'/L.

Let (ℓ_n) be a sequence representing $\mathfrak{u}$ such that $\mathrm{Frob}_{\ell_n}\ |_{L'/K}= c$, for some $c \in \mathrm{Gal}(L'/K)/\sim$ for $\mathcal{U}$-many n. Then $\mathrm{Frob}_{\mathfrak{u}}\ |_{L'/K}$ is the conjugacy class c. For those n, $\mathrm{Frob}_{\ell_n}\ |_L$, coincides with the image of $\mathrm{Frob}_{\ell_n}\ |_{L'/K}$ under the projection to $\mathrm{Gal}(L/K)$. Since it holds for $\mathcal{U}$-many n, then $\mathrm{Frob}_{\mathfrak{u}}\ |_{L/K}$ coincides with the projection of $\mathrm{Frob}_{\mathfrak{u}}\ |_{L'/K}$ to $\mathrm{Gal}(L/K)$. Therefore, the inverse limit in Definition 4.1.29 is well-defined and describes a conjugacy class in G_K.

Remark 4.1.31. Definition 4.1.29 is consistent with the standard definition of Frobenius automorphisms: if $\mathfrak{u} = (\ell)$ is a constant ultraprime, then $\mathrm{Frob}_{\mathfrak{u}} = \mathrm{Frob}_\ell$.

Ultraprimes have their own version of the Chebotarev density theorem, which is stronger than the classical version. Its main advantage is that it is no longer restricted to finite extensions. The following result is proven in [LZ25].

Proposition 4.1.32. ([LZ25, Lemma 5.4]) Let K be a number field and let $c \subset G_K$ be a conjugacy class. Then there exists an ultraprime $\mathfrak{u}$ such that $\mathrm{Frob}_{\mathfrak{u}} = c$.

Proof. Let $(L_n)_{n\in\mathbb{N}}$ be an ordering of all the finite extensions of K. For every $n \in \mathbb{N}$, define $E_n = L_1 \cdots L_n$. By the Chebotarev density theorem, there exists a prime ℓ_n such that $\mathrm{Frob}_{\ell_n} = c|_{E_n}$.

Consider the prime $\mathfrak{u} = (\ell_n)$. For every number field L, there exists a natural number n_0 such that $L = L_{n_0}$. Then $\mathrm{Frob}_{\ell_n} |_L = c |_L$ for all $n \geq n_0$ and, therefore, for $\mathcal{U}$-many n. Hence $\mathrm{Frob}_{\mathfrak{u}} |_L = c |_L$ for all number fields L, so $\mathrm{Frob}_{\mathfrak{u}} = c$. $\square$

The construction of the Frobenius is used to artificially define the local Galois group at the ultraprime. It is a generalization of the tame quotient of the classical local Galois groups.

Definition 4.1.33. Let $\mathfrak{u}$ be a non-constant ultraprime. The *local Galois group* $G_{\mathfrak{u}}$ is defined as the semidirect product

$$G_{\mathfrak{u}} := \widehat{\mathbb{Z}}(1) \rtimes \langle \mathrm{Frob}_{\mathfrak{u}} \rangle$$

where $\langle \mathrm{Frob}_{\mathfrak{u}} \rangle$ is the free profinite group generated by one element which acts by $\mathrm{Frob}_{\mathfrak{u}} \in G_K$ on $\widehat{\mathbb{Z}}(1)$. The *inertia subgroup* $I_{\mathfrak{u}} \subset G_{\mathfrak{u}}$ is the normal subgroup $\widehat{\mathbb{Z}}(1)$.

Remark 4.1.34. Note that the action of $\mathrm{Frob}_{\mathfrak{u}}$ on $\widehat{Z}(1)$ is well-defined independently of the choice of the Frobenius element in the conjugacy class, since $K(\mu_{\mathfrak{p}^\infty})/K$ is an abelian extension.

Remark 4.1.35. Note that, when $\mathfrak{u}$ is a constant ultraprime, the semidirect product $\widehat{\mathbb{Z}}(1) \rtimes \langle \mathrm{Frob}_{\mathfrak{u}} \rangle$ coincides with the tame inertia quotient of the Galois group $G_{\mathfrak{u}}$.

We impose that $G_{\mathfrak{u}}$ acts unramifiedly on Galois modules.

Definition 4.1.36. Let T be a G_K-module and pick a Frobenius element $\mathrm{Frob}_{\mathfrak{u}}$. We define an action of $G_{\mathfrak{u}}$ on T via the quotient $G_{\mathfrak{u}} \twoheadrightarrow \langle \mathrm{Frob}_{\mathfrak{u}} \rangle$.

Remark 4.1.37. The action of $G_{\mathfrak{u}}$ on T is only well defined up to conjugation, since it involves the choice of the Frobenius element.

However, that is no different than the actions of classical Frobenius elements, since they were only defined up to conjugation.

4.1.5 Product of ultraprimes

It is possible to define products of ultraprimes, obtaining classes in $\mathcal{U}(\mathcal{I}_K)$, where $\mathcal{I}_K$ denotes the set of ideals of K. Indeed, for every $s \in \mathbb{N}$, there is a well defined map

$$\mathcal{U}(\mathbb{P}_K)^s \to \mathcal{U}(\mathcal{I}_K) : \ (\mathfrak{u}_1, \ldots, \mathfrak{u}_s) \mapsto \mathfrak{u}_1 \cdots \mathfrak{u}_s$$

This product satisfies a weaker analogue of the unique prime factorisation theorem on $\mathcal{I}_K$, since the factorizations are unique but not all of the elements in $\mathcal{U}(\mathcal{I}_K)$ admit an ultraprime factorization.

Since the product of ultraprimes is commutative, we can express it in the following way.

Definition 4.1.38. The product of ultraprimes is a map

$$\mathcal{U}(\mathbb{P}_K)^s / S_s \to \mathcal{U}(\mathcal{I}_K)$$

where S_s denotes the symmetric group, acting in the natural way. The map is defined as follows: for every finite set of ultraprimes $\mathfrak{u}_1 = \left(\ell_k^{(1)}\right)_{k\in\mathbb{N}}, \ldots, \mathfrak{u}_s = \left(\ell_k^{(s)}\right)_{k\in\mathbb{N}}$, we define

$$\mathfrak{u}_1 \cdots \mathfrak{u}_s = (\ell_n^1 \cdots \ell_n^{(s)})_{n\in\mathbb{N}} \in \mathcal{U}(\mathcal{I}_K)$$

Remark 4.1.39. Definition 4.1.38 produces a well-defined class in $\mathcal{U}(\mathcal{I}_K)$, which is represented by a sequence $(n_i)_{i\in\mathbb{N}}$ in which $\nu(n_i) = s$ for all $i \in \mathbb{N}$, where $\nu(I)$ denotes the number of prime ideals dividing I, counting multiplicities. Hence, the product of s ultraprimes cannot be equal to the product of s' ultraprimes.

We can show that the map in Definition 4.1.38 is injective.

Proposition 4.1.40. The product in Definition 4.1.38 induces an injective map

$$\bigsqcup_{s>0} : \ \mathcal{U}(\mathbb{P}_K)^s / S_s \to \mathcal{U}(\mathcal{I}_K)$$

Proof. Assume that $\mathfrak{n} = \mathfrak{u}_1 \cdots \mathfrak{u}_s = \mathfrak{q}_1 \cdots \mathfrak{q}_{s'} \in \mathcal{U}(\mathcal{I}_K)$. By Remark 4.1.39, we have that $s = s'$. Choose representatives

$$\mathfrak{u}_i = \left(\ell_k^{(i)}\right)_{k\in\mathbb{N}}, \quad \mathfrak{q}_j = \left(q_k^{(j)}\right)_{k\in\mathbb{N}}$$

By the definition of $\mathcal{U}(\mathcal{I}_K)$

$$\ell_k^{(1)} \cdots \ell_k^{(s)} = q_k^{(1)} \cdots q_k^{(s)} \text{ for } \mathcal{U}\text{-many } k.$$

By Corollary 4.1.3 and the unique factorization in $\mathcal{I}_K$, for every $i \leq s$, there is some $j \leq s$ such that $\ell_k^{(i)} = q_k^{(j)}$ for $\mathcal{U}$-many k, leading to $\mathfrak{u}_i = \mathfrak{q}_j$. This shows that $(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ is a reordering of $(\mathfrak{u}_1, \ldots, \mathfrak{u}_s)$. $\square$

Remark 4.1.39 and Proposition 4.1.40 prove the uniqueness of the ultraprime factorization. Now it makes sense to mention square-free products of ultraprimes.

Definition 4.1.41. A product of ultraprimes is called *square-free* if no two $\mathfrak{u}_i$ are equal.

As mentioned before, the ultraprime factorisation does not exists for every $\mathfrak{n} \in \mathcal{U}(\mathcal{I}_K)$, but only when it can be represented by a sequence $(n_k)_{k\in\mathbb{N}}$ in which the number of prime divisors of n_k, counting multiplicities, remains bounded.

Proposition 4.1.42. Assume an element in $\mathfrak{n} \in \mathcal{U}(\mathcal{I}_K)$ can be represented by a sequence $(n_k)_{k\in\mathbb{N}}$ such that $\nu(n_k)$ is bounded. Then $\mathfrak{n}$ can be obtained as a product of ultraprimes.

Assume, in addition, that n_k is a square-free product for $\mathcal{U}$-many k, then $\mathfrak{n}$ is also a square-free product.

Proof. By Corollary 4.1.3, there exists $s \in \mathbb{N}$ and $S \in \mathcal{U}$ such that $\nu(n_k) = s$ for all $k \in S$. It is possible to choose s sequences of prime numbers $\left(\ell_k^{(j)}\right)_{i\in\mathbb{N}}$ such that, whenever $k \in S$, then

$$n_k = \ell_k^{(1)} \cdots \ell_k^{(s)}$$

By construction, $\mathfrak{n} = \mathfrak{u}_1 \cdots \mathfrak{u}_s$, where $\mathfrak{u}_j$ is the ultraprime represented by the sequence $\left(\ell_k^{(j)}\right)_{k\in\mathbb{N}}$.

Note that, if $\mathfrak{u}_i = \mathfrak{u}_j$ for some i and j, every representative $(n_i)_{i\in\mathbb{N}}$ of $\mathfrak{u}_1 \cdots \mathfrak{u}_s$ will not contain $\mathcal{U}$-many square-free n_i, so the last statement holds. $\square$

4.2 Patched cohomology

4.2.1 Construction

In this section, we use the ultraproduct defined in the previous one to introduce the notion of patched cohomology. It is the base to extend the classical theory of local and global Galois cohomology to the ultraprimes defined in the previous section.

This section does not contain any new result, but it is a survey of the results in [Swe22, §2], where the notion of patched cohomology was first introduced.

In order to have control over the patched cohomology groups, we must define if first for finite coefficient rings as an ultraproduct of cohomology groups, and then we extend the definition to either profinite or ind-finite coefficient rings by taking limits.

Definition 4.2.1. Let T be a finite, abelian group endowed with actions from a sequence[2] of groups $G = (G_n)_{n\in\mathbb{N}} \in \mathcal{U}(\{\text{groups}\})$. The $\mathcal{U}$-patched cohomology group is defined as

$$\mathbf{H}^i(G,T) := \mathcal{U}(H^i(G_n,T))$$

If T is a profinite abelian group, we define the patched cohomology as

$$\mathbf{H}^i(G,T) := \varprojlim_{T\twoheadrightarrow T'} \mathbf{H}^i(G,T/T')$$

where the limit is taken over all the finite quotients of T.

Similarly, when T is an ind-finite abelian group, the patched cohomology is defined as

$$\mathbf{H}^i(G,T) := \varinjlim_{T'\hookrightarrow T} \mathbf{H}^i(G,T')$$

where the limit is taken over all the finite subgroups of T.

Proposition 4.2.2. The assignment

$$T \mapsto \mathbf{H}^i(G,T)$$

is a functor from the category of either finite abelian groups, profinite abelian groups and ind-finite abelian groups to the category of abelian groups.

Proof. It follows from the functorial properties of cohomology, ultraproducts, and inverse and direct limits. $\square$

When the coefficient group T is finite, there is a long exact sequence analogously to classical group cohomology.

[2]Technically, it is only needed that the action is well defined for $\mathcal{U}$-many n

Proposition 4.2.3. Let

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C \longrightarrow 0$$

be an exact sequence of finite abelian groups. Assume A, B and C are endowed with an action of $G = (G_n)$. Then there is an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{H}^i(G,A) \arrow[r] & \mathbf{H}^i(G,B) \arrow[r] & \mathbf{H}^i(G,C) \arrow[dll, out=-10, in=170] \\
& \mathbf{H}^{i+1}(G,A) \arrow[r] & \mathbf{H}^{i+1}(G,B) \arrow[r] & \mathbf{H}^{i+1}(G,C) \arrow[dll, out=-10, in=170] \\
& \mathbf{H}^{i+2}(G,A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Since A, B and C are finite, we have that

$$\mathbf{H}^i(G,-) = \mathcal{U}_n\big(H^i(G_n,-)\big)$$

Then the result follows from Proposition 4.1.20 and the classical long exact sequence for Galois cohomology. $\square$

Under the following assumption, we can construct a long cohomological exact sequence for the patched cohomology with profinite coefficients.

Definition 4.2.4. We say a sequence of groups $G = (G_n)$ satisfies the *bounded cohomology property* for some index $i \in \mathbb{Z}_{\geq 0}$ if for every finite, abelian group T endowed with an action of G, there is a constant A such that

$$H^i(G_n,T) \leq A \text{ for } \mathcal{U}\text{-many } n$$

Remark 4.2.5. The bounded cohomology property guarantees, using Corollary 4.1.22, that the patched cohomology group $\mathbf{H}^1(G,T)$ is finite for all finite abelian groups T.

Proposition 4.2.6. Let

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C \longrightarrow 0$$

be an exact sequence of continuous maps of profinite abelian groups. Assume A, B and C are endowed with an action of $G = (G_n)$, where G_n satisfies the bounded cohomology property for all $i \in \mathbb{Z}_{\geq 0}$. Then there is an exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbf{H}^i(G,A) \arrow[r] & \mathbf{H}^i(G,B) \arrow[r] & \mathbf{H}^i(G,C) \arrow[dll, out=-5, in=175] \\
& \mathbf{H}^{i+1}(G,A) \arrow[r] & \mathbf{H}^{i+1}(G,B) \arrow[r] & \mathbf{H}^{i+1}(G,C) \arrow[dll, out=-5, in=175] \\
& \mathbf{H}^{i+2}(G,A) \arrow[r] & \cdots &
\end{tikzcd}
$$

Proof. Let I be a directed set indexing the finite quotients of B, i.e., all the finite quotients of B are of the form B/β_i, for some $i \in I$. Since B is profinite, we have that

$$\bigcap_{i \in I} \beta_i = 0$$

Since μ is injective,

$$\bigcap_{i \in I} \mu^{-1}(\beta_i) = 0$$

We can use this fact to show that $\mu^{-1}(\beta_i)$ is cofinal. Indeed, let A' be a finite index subgroup of A. Then

$$\bigcap_{i \in I} \Big((A \setminus A') \cap \mu^{-1}(\beta_i) \Big) = \emptyset$$

Since $A \setminus A'$ is compact and the set $\{\mu^{-1}(\beta_i) : i \in I\}$ is closed under finite intersections, there exists some $i \in I$ such that

$$(A \setminus A') \cap \mu^{-1}(\beta_i) = \emptyset$$

Equivalently, $\mu^{-1}(\beta_i) \subset A'$. Hence the patched cohomology of A can be computed as the inverse limit

$$\mathbf{H}^i(G,A) = \varprojlim_{A \twoheadrightarrow A'} \mathbf{H}^i(G,A') = \varprojlim_{i \in I} \mathbf{H}^i(G, A/\mu^{-1}(\beta_i))$$

Similarly, $\{\varepsilon(\beta_i) : i \in I\}$ is cofinal since, for every open subgroup $C' \subset C$, $\varepsilon^{-1}(C')$ has finite index and $C' = \varepsilon\big(\varepsilon^{-1}(C')\big)$. Thus,

$$\mathbf{H}^i(G,C) = \varprojlim_{C \twoheadrightarrow C'} \mathbf{H}^i(G,C/C') = \varprojlim_{i \in I} \mathbf{H}^i(G, C/\varepsilon(\beta_i))$$

For each $i \in I$, there is an exact sequence

$$0 \longrightarrow A/\mu^{-1}(\beta_i) \xrightarrow{\mu} B/\beta_i \xrightarrow{\varepsilon} C/\varepsilon(\beta_i) \longrightarrow 0$$

By Proposition 4.2.3, there is a long exact sequence

$$\begin{aligned}
\cdots \longrightarrow \mathbf{H}^i\left(G, \tfrac{A}{\mu^{-1}(\beta_i)}\right) &\longrightarrow \mathbf{H}^i\left(G, \tfrac{B}{\beta_i}\right) \longrightarrow \mathbf{H}^i\left(G, \tfrac{C}{\varepsilon(\beta_i)}\right) \longrightarrow \\
\longrightarrow \mathbf{H}^{i+1}\left(G, \tfrac{A}{\mu^{-1}(\beta_i)}\right) &\to \mathbf{H}^{i+1}\left(G, \tfrac{B}{\beta_i}\right) \to \mathbf{H}^{i+1}\left(G, \tfrac{C}{\varepsilon(\beta_i)}\right) \longrightarrow \\
\longrightarrow \mathbf{H}^{i+2}\left(G, \tfrac{A}{\mu^{-1}(\beta_i)}\right) &\longrightarrow \cdots
\end{aligned}$$

The bounded cohomology property of the sequence G implies that all the groups in this exact sequence are finite (see Remark 4.2.5). Since the inverse limit of finite groups is an exact functor, the long cohomological sequence is exact. $\square$

4.2.2 Local patched cohomology

In this section, we outline the basic properties of the local cohomology at an ultraprime $\mathfrak{u}$, defined as the patching of the local cohomology of the primes defining $\mathfrak{u}$.

Definition 4.2.7. Let $\mathfrak{u}$ be an ultraprime represented by the sequence (ℓ_n). The local patched cohomology group is defined as

$$\mathbf{H}^i(K_\mathfrak{u}, T) := \mathbf{H}^i\big((G_{K_{\ell_n}}), T\big)$$

Remark 4.2.8. Local patched cohomology satisfies the bounded cohomology property in Definition 4.2.4 since, for every finite abelian group T,

$$\#H^1(K_\ell, T) \leq 2\#T$$

for all primes ℓ not dividing $\#T$. Then Proposition 4.2.6 proves the existence of a long exact sequence when the coefficients form a profinite group.

The local patched cohomology coincides with the group cohomology of the local Galois group defined in Definition 4.1.33.

Proposition 4.2.9. ([Swe22, Proposition 2.3.2]) Let T be either a finite, profinite or ind-finite abelian group, and let $\mathfrak{u}$ be an ultraprime represented by the sequence (ℓ_n). Then

$$\mathbf{H}^i(K_{\mathfrak{u}}, T) = \mathbf{H}^i(G_{\mathfrak{u}}, T)$$

where $G_{\mathfrak{u}}$ is the patched local Galois group defined in Definition 4.1.33.

Proof. By taking limits, we only need to prove it when T is finite. If $\mathfrak{u}$ is a constant ultraprime, it follows from the finiteness of classical local Galois cohomology and Lemma 4.1.21.

Hence we can assume that $\mathfrak{u}$ is a non-constant ultraprime. For $\mathcal{U}$-many n, the action of G_{ℓ_n} on T is unramified, Frob_{ℓ_n} acts on T like $\mathrm{Frob}_{\mathfrak{u}}$ and $\ell_n \nmid \#T$. Therefore, for those indices, we have that

$$H^0(G_{\mathfrak{u}}, T) = H^0(G_{l_n}, T)$$

For those values, we also have that

$$H^i(G_{\ell_n}, T) = H^i(G^{\mathrm{t}}_{\ell_n}, T)$$

where $G^{\mathrm{t}}_{\ell_n}$ is Galois group of the maximal tamely ramified extension. The tame Galois group of K_{ℓ_n} can be identified with the direct product

$$G^t_{\ell_n} = \mathcal{I}^t_{\ell_n} \rtimes \langle \mathrm{Frob}_{\ell_n} \rangle = \prod_{q \neq \ell_n} \mathbb{Z}_q(1) \rtimes \langle \mathrm{Frob}_{\ell_n} \rangle$$

The isomorphism $\mathcal{I}^t_{\ell_n} \times \mathbb{Z}_{\ell_n}(1) \cong \widehat{\mathbb{Z}}(1)$ induces an action of Frob_{ℓ_n} on $\widehat{\mathbb{Z}}(1)$. By construction, the actions of Frob_{ℓ_n} and $\mathrm{Frob}_{\mathfrak{u}}$ coincide on every finite quotient of $\mathbb{Z}(1)$ for $\mathcal{U}$-many n.

Note that both $G_{\mathfrak{u}}$ and G_{ℓ_n} have cohomological dimension two, so we only need to prove the proposition for $i \leq 2$. We start with $i = 1$. For every $q = \mathfrak{u}, \ell_1, \ldots, \ell_n, \ldots$, there is an inflation-restriction map

$$H^1\left(\frac{\mathcal{I}_q}{(\#T)\mathcal{I}_q} \rtimes \langle \mathrm{Frob}_q \rangle, T\right) \rightarrowtail H^1(G_q, T) \longrightarrow H^1((\#T)\mathcal{I}_q, T)$$

However, the restriction map is the zero map since it factors trough $H^1(I_q, T)$, so the first map in the previous exact sequence is an isomorphism:

$$H^1(G_{\mathfrak{q}}, T) \cong H^1\left(\frac{\mathcal{I}_{\mathfrak{q}}}{\#T\mathcal{I}_{\mathfrak{q}}} \rtimes \langle \mathrm{Frob}_{\mathfrak{q}} \rangle, T\right)$$

By construction, for $\mathcal{U}$-many n, there is an isomorphisms

$$H^1\left(\frac{\mathcal{I}_{\mathfrak{u}}}{\#T\mathcal{I}_{\mathfrak{u}}} \rtimes \langle \mathrm{Frob}_{\mathfrak{u}} \rangle, T\right) \cong H^1\left(\frac{\mathcal{I}_{\ell_n}}{\#T\mathcal{I}_{\ell_n}} \rtimes \langle \mathrm{Frob}_{\ell_n} \rangle, T\right)$$

Therefore,

$$H^1(G_{\mathfrak{u}}, T) \cong H^1(G_{\ell_n}, T)$$

For $i = 2$, the result follows by a similar argument using the following identity arising from Hoschild-Serre spectral sequence:

$$H^2(G_q, T) \cong H^1\big(\langle \mathrm{Frob}_q \rangle, H^1(\mathcal{I}_q, T)\big)$$

Then,

$$H^2(G_q, T) \cong H^1\left(\langle \mathrm{Frob}_q \rangle, \mathrm{Hom}\left(\frac{\mathcal{I}_q}{(\#T)\mathcal{I}_q}, T\right)\right) \qquad \square$$

From this proof, we obtain the following corollary.

Corollary 4.2.10. Let $\mathfrak{u} = (\ell_n)$ be an ultraprime and let T be a finite group endowed with an action of G_K. For $\mathcal{U}$-many i, there is an isomorphism

$$\varphi_i^{\mathfrak{u}}: \mathbf{H}^1(K_{\mathfrak{u}}, T) \cong H^1(K_{\ell_i}, T)$$

Definition 4.2.11 (Cohomology of the inertia subgroup)**.** Let $\mathfrak{u}$ be an ultraprime represented by the sequence (ℓ_n). The cohomology of the inertia subgroup $\mathcal{I}_{\mathfrak{u}}$ is defined as the patched cohomology group

$$\mathbf{H}^i(\mathcal{I}_{\mathfrak{u}}, T) := \mathbf{H}^i\big(\mathcal{I}_{\ell_n}, T\big)$$

The notation suggests we can obtain the patched cohomology as the group cohomology of the inertia group $\mathcal{I}_{\mathfrak{u}}$ defined in Definition 4.1.33.

Proposition 4.2.12. Let $\mathfrak{u}$ be an ultraprime represented by a sequence $(\ell_n)_{n\in\mathbb{N}}$. The patched cohomology of the inertia subgroups $\mathcal{I}_{\ell_n}$ coincides with the group cohomology of the inertia subgroup $\mathcal{I}_{\mathfrak{u}}$:

$$\mathbf{H}^i\big(\mathcal{I}_{\ell_n}, T\big) = H^i(\mathcal{I}_{\mathfrak{u}}, T)$$

Proof. Similarly to Proposition 4.2.9, we can assume that T is finite. For $\mathcal{U}$-many n, the action of G_{ℓ_n} is unramified and $\ell_n \nmid \#T$. For those n, we have that

$$H^i(\mathcal{I}_{\mathfrak{u}}, T) = H^1(\mathcal{I}_{\ell_n}, T) = \mathcal{U}_n\big(H^i(\mathcal{I}_{\ell_n}, T)\big) \qquad \square$$

Similarly as in the local patched cohomology, we can obtain the following corollary.

Corollary 4.2.13. Let $\mathfrak{u} = (\ell_n)$ be an ultraprime and let T be a finite group endowed with an action of G_K. For $\mathcal{U}$-many i, there is an isomorphism

$$\varphi_i^{\mathcal{I}_{\mathfrak{u}}} : \; \mathbf{H}^1(\mathcal{I}_{\mathfrak{u}}, T) \cong H^1(\mathcal{I}_{\ell_i}, T)$$

Definition 4.2.14 (Finite cohomology)**.** The *finite patched cohomology group* is defined as

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) = \ker\Big(\mathbf{H}^1(K_{\mathfrak{u}}, T) \to \mathbf{H}^1(I_{\mathfrak{u}}, T)\Big)$$

Definition 4.2.15 (Singular cohomology)**.** We define the *singular patched cohomology group* as

$$\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) = \frac{\mathbf{H}^1(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)}$$

Remark 4.2.16. The finite patched cohomology groups can be obtain as the patching of the classical finite cohomology groups. Indeed, when T is finite, Proposition 4.1.20 implies that

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) = \mathcal{U}_i\big(H^1_{\mathrm{f}}(K_{\ell_i}, T)\big)$$

When T is profinite (resp. ind-finite), the left exactness of the inverse limit (resp. direct limit) implies that $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$ can be obtained as the inverse limit (resp. direct limit) of the finite patched cohomology $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T')$ of the finite quotients (resp. finite submodules) T' of T.

From Corollaries 4.2.10 and 4.2.13, we can obtain the following result.

Corollary 4.2.17. Let $\mathfrak{u} = (\ell_n)$ be an ultraprime and let T be a finite group endowed with an action of G_K. For $\mathcal{U}$-many n, there is an isomorphism

$$\varphi_i^{\mathfrak{u},\mathrm{f}}:\ \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \cong H^1_{\mathrm{f}}(K_{\ell_i}, T)$$

Local Tate duality extend to patched cohomology.

Proposition 4.2.18. (Local duality) Let T be either a finite, profinite or ind-finite abelian group and let $\mathfrak{u} = (\ell_n)$ be an ultraprime, which is not a constant archimedean prime. For $i = 0, 1, 2$, there is a non-degenerate pairing

$$\mathbf{H}^i(K_{\mathfrak{u}}, T) \times \mathbf{H}^{2-i}(K_{\mathfrak{u}}, T^*) \to \mathbb{Q}/\mathbb{Z}$$

Proof. Assume first that T is finite. Then

$$\mathbf{H}^i(K_{\mathfrak{u}}, T) = \mathcal{U}\big(H^i(K_{\ell_n}, T)\big),\quad \mathbf{H}^{2-i}(K_{\mathfrak{u}}, T^*) = \mathcal{U}\big(H^{2-i}(K_{\ell_n}, T^*)\big)$$

Hence, by Proposition 4.1.23

$$\mathbf{H}^i(K_{\mathfrak{u}}, T) \times \mathbf{H}^{2-i}(K_{\mathfrak{u}}, T^*) = \mathcal{U}_n\Big(H^i(K_{\ell_n}, T) \times H^{2-i}(K_{\ell_n}, T^*)\Big)$$

Classical Tate duality, together with the functoriality of the ultraproduct, induces a pairing

$$\mathbf{H}^i(K_{\mathfrak{u}}, T) \times \mathbf{H}^{2-i}(K_{\mathfrak{u}}, T^*) \to \mathcal{U}\left(\frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right) = \frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}$$

Since classical local duality is non-degenerate, and the ultraproduct is an exact functor, there is an injection

$$\mathcal{U}_n\big(H^i(K_{\ell_n}, T)\big) \hookrightarrow \mathcal{U}_n\left(\mathrm{Hom}\left(H^{2-i}(K_{\ell_n}, T^*), \frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right)$$

Applying Proposition 4.1.24, we also obtain the injection

$$\mathcal{U}_n\big(H^i(K_{\ell_n}, T)\big) \hookrightarrow \mathrm{Hom}\left(\mathcal{U}_n\big(H^{2-i}(K_{\ell_n}, T^*)\big), \mathcal{U}\left(\frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right)$$

Therefore, the patched local duality pairing is non-degenerate on the left. Similarly, it is also non-degenerate on the right factor.

Assume now that T is profinite, which implies that T^* is ind-finite. Then

$$\begin{aligned}\mathbf{H}^i(K_{\mathfrak{u}}, T) &= \varprojlim_{T \twoheadrightarrow T'} \mathbf{H}^i(K_{\mathfrak{u}}, T') \\ \mathbf{H}^{2-i}(K_{\mathfrak{u}}, T^*) &= \varinjlim_{T \twoheadrightarrow T'} \mathbf{H}^{2-i}(K_{\mathfrak{u}}, (T')^*)\end{aligned}$$

Hence, local duality for patched cohomology groups with finite coefficients induces a perfect pairing

$$\varprojlim_{T \twoheadrightarrow T'} \mathbf{H}^i(K_{\mathfrak{u}}, T') \times \varinjlim_{T \twoheadrightarrow T'} \mathbf{H}^{2-i}(K_{\mathfrak{u}}, (T')^*) \to \mathbb{Q}/\mathbb{Z}$$

It proves local duality for profinite T. When T is ind-finite, an analogous proof completes the result. $\square$

Proposition 4.2.19. (Local duality) Let T be either a finite, profinite or ind-finite group and let $\mathfrak{u} = (\ell_n)$ be an ultraprime. Under the duality pairing, $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$ and $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T^*)$ are annihilators of each other.

Proof. When T is finite, the classical result provides an isomorphism

$$\mathbf{H}^1_{\mathrm{f}}(K_{\ell_i}, T) = \ker\left(\mathbf{H}^1(K_{\ell_i}, T) \to \mathrm{Hom}\left(\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T^*), \frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right)$$

The patched analogue follows from Proposition 4.1.20, the exactness of inverse and direct limits of finite groups and the application of Proposition 4.1.24 as in the proof of Proposition 4.2.18. $\square$

4.2.3 Global patched cohomology

The goal of this section is to define global patched cohomology groups. We do this by generalising the notion of Galois cohomology of the maximal extension K^{Σ}/K unramified outside the certain finite set of prime numbers Σ. Hence, we aim to construct the patched cohomology group unramified outside a finite set of ultraprimes.

Definition 4.2.20. ([Swe22, Definition 2.4.2]) Let T be either a finite, profinite or ind-finite abelian group and let $\mathfrak{n} \in \mathcal{N}$ be a squarefree product of ultraprimes represented by the sequence $(n_k)_{k \in \mathbb{N}}$. We define the maximal patched cohomology group unramified at $\mathfrak{n}$ by

$$\mathbf{H}^i(K^{\mathfrak{n}}/K, T) := \mathbf{H}^i\big(\mathrm{Gal}(K^{n_k}/K), T^{G_{K_{n_k}}}\big)$$

where K^{n_k} represents the maximal extension of K unramified outside the prime divisors of n_k. Note that this definition is independent of the sequence representing $\mathfrak{n}$.

Notation 4.2.21. If Σ is a finite set of distinct ultraprimes and $\mathfrak{n}$ is the product of all the ultraprimes in Σ, we will also denote $\mathbf{H}^i(K^{\mathfrak{n}}/K, T)$ by $\mathbf{H}^i(K^{\Sigma}/K, T)$.

The key technical property of the global patched cohomology groups is the bounded cohomology property.

Proposition 4.2.22. ([Swe22, Lemma 2.4.5]) Assume that $\mathfrak{n} \in \mathcal{U}(\mathcal{I}_K)$ is a finite product of ultraprimes. For every sequence $(n_i)_{i\in\mathbb{N}}$ representing $\mathfrak{n}$, the sequence

$$G(K^{\mathfrak{n}}/K) := (\mathrm{Gal}(K^{n_i}/K))_{i\in\mathbb{N}}$$

satisfies the bounded cohomology property stated in Definition 4.2.4.

Proof. Say that $\mathfrak{n}$ is the product of S ultraprimes. Then $\nu(n_i) = s$ for $\mathcal{U}$-many i. Then the proposition follows from [Swe22, Lemma 2.4.5] (see also [Mil06]). $\square$

Corollary 4.2.23. Let T be a finite group and let $\Sigma \subset \mathcal{U}(\mathbb{P}_K)$ be a finite set of ultraprimes. The patched cohomology groups $\mathbf{H}^i(K^{\Sigma}/K, T)$ are finite for all $i \geq 0$.

Since the bounded cohomology property holds, we can apply Proposition 4.2.6 to obtain the following result.

Corollary 4.2.24. ([Swe22, Lemma 2.4.5]) Consider the following exact sequence of profinite abelian groups:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Then there is a long exact sequence of global patched cohomology groups

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{H}^i(K^{\Sigma}/K, A) \arrow[r] & \mathbf{H}^i(K^{\Sigma}/K, B) \arrow[r] & \mathbf{H}^i(K^{\Sigma}/K, C) \arrow[dll] \\
& \mathbf{H}^{i+1}(K^{\Sigma}/K, A) \arrow[r] & \mathbf{H}^{i+1}(K^{\Sigma}/K, B) \arrow[r] & \mathbf{H}^{i+1}(K^{\Sigma}/K, C) \arrow[dll] \\
& \mathbf{H}^{i+2}(K^{\Sigma}/K, A) \arrow[r] & \cdots &
\end{tikzcd}$$

Remark 4.2.25. If $\Sigma_1 \subset \Sigma_2 \subset \mathcal{U}(\mathbb{P}_K)$ are two finite sets of ultraprimes. Then there is a natural map

$$\mathbf{H}^i(K^{\Sigma_1}/K, T) \hookrightarrow \mathbf{H}^i(K^{\Sigma_2}/K, T)$$

When T is finite, it is induced by a sequence of inflation maps. The general case, when T is profinite and ind-finite, follows by taking limits.

We can use the maps in Remark 4.2.25 to construct the absolute global patched cohomology group.

Definition 4.2.26. ([Swe22, §2.4.6]) The absolute global patched cohomology group is defined as

$$\mathbf{H}^1(K,T) := \varinjlim_{\Sigma \subset \mathcal{U}(\mathbb{P}_K)} \mathbf{H}(K^{\Sigma}/K,T)$$

where the limit is taken over all the finite sets and the transition maps are the ones defined in Remark 4.2.25.

Definition 4.2.27. Let Σ be a finite set of ultraprimes and let $\mathfrak{u}$ be an ultraprime. There exists a restriction map

$$\mathrm{res}: \ \mathbf{H}^1(K^{\Sigma}/K,T) \to \mathbf{H}^1(K_{\mathfrak{u}},T)$$

induced, when T is finite, by the restriction map in every factor of the ultraproduct. When T is profinite (resp. ind-finite), the restriction map is obtained as the limit of the restriction map in the cohomology of the finite quotients (resp. submodules).

We now show that the above definition is, in fact, the unramified subgroup of the global patched cohomology.

Proposition 4.2.28. ([Swe22, Proposition 2.4.11]) Let $\Sigma \subset \mathcal{U}(\mathbb{P}_K)$ be a finite set of ultraprimes. Then

$$\mathbf{H}^1(K^{\Sigma}/K,T) = \ker\left(\mathbf{H}^1(K,T) \to \prod_{\mathfrak{u} \in \mathcal{U}(\mathbb{P}_K)\setminus\Sigma} \frac{\mathbf{H}^1(K_{\mathfrak{u}},T)}{\mathbf{H}^1(\mathcal{I}_{\mathfrak{u}},T)}\right)$$

Proof. By the Definition 4.2.26 and the exactness of the direct limit, it suffices to show that, whenever Σ' is another finite set of ultraprimes, disjoint to Σ, then

$$\mathbf{H}^1(K^{\Sigma}/K,T) = \ker\left(\mathbf{H}^1(K^{\Sigma\cup\Sigma'}/K,T) \to \bigoplus_{\mathfrak{u}\in\Sigma'} \frac{\mathbf{H}^1(K_{\mathfrak{u}},T)}{\mathbf{H}^1(\mathcal{I}_{\mathfrak{u}},T)}\right)$$

When T is finite, it follows by the exactness of the ultraproduct (see Proposition 4.1.20). In other cases, it follows from the left exactness of inverse and direct limits. $\square$

Local and glocal patched cohomology also satisfy an analogue of Poitou-Tate 9-term exact sequence.

Proposition 4.2.29. Let Σ be a finite set of ultraprimes and let T be either a finite, profinite or ind-finite group endowed with an action of G_K. Then there is a 9-term exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^0(K^\Sigma/K, T) \arrow[r] & \prod_{\mathfrak{u}\in\Sigma} \mathbf{H}^0(K_\mathfrak{u}, T) \arrow[r] & \mathbf{H}^2(K^\Sigma/K, T^*)^\vee \arrow[dll] & \\
 & \mathbf{H}^1(K^\Sigma/K, T) \arrow[r] & \prod_{\mathfrak{u}\in\Sigma} \mathbf{H}^1(K_\mathfrak{u}, T) \arrow[r] & \mathbf{H}^1(K^\Sigma/K, T^*)^\vee \arrow[dll] & \\
 & \mathbf{H}^2(K^\Sigma/K, T) \arrow[r] & \prod_{\mathfrak{u}\in\Sigma} \mathbf{H}^0(K_\mathfrak{u}, T^*)^\vee \arrow[r] & \mathbf{H}^0(K^\Sigma/K, T^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

Proof. When T is finite, it follows from Poitou-Tate duality in classical Galois cohomology ([NSW00, Theorem 8.4.4]) and the exactness of the ultraproduct in Proposition 4.1.20. When T is either profinite or ind-finite, it follows by taking limits. $\square$

4.3 Patched Selmer groups

4.3.1 Patched Selmer structures

Following [Swe22], we can now define the Selmer structures in this setting. The main innovation is that they also include local conditions at non-constant ultraprimes. However, classical Selmer structures can be recovered when the local condition is the finite cohomology group for every non-constant ultraprime.

Definition 4.3.1. A *Selmer structure* $\mathcal{F}$ consists of the following data:

- A finite set $\Sigma_\mathcal{F}$ of $\mathcal{U}(\mathbb{P}_K)$ containing all constant ultraprimes lying over p, ∞ and ramified (constant) ultraprimes of T.
- For each $\mathfrak{u} \in \Sigma_\mathcal{F}$, a closed R-submodule

$$\mathbf{H}^1_\mathcal{F}(K_\mathfrak{u}, T) \subset \mathbf{H}^1(K_\mathfrak{u}, T)$$

Definition 4.3.2. The Selmer group of a Selmer structure $\mathcal{F}$ is defined as

$$\mathbf{H}^1_{\mathcal{F}}(K,T) := \ker\left(\mathbf{H}^1(K^{\Sigma_{\mathcal{F}}}/K,T) \to \prod_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \frac{\mathbf{H}^1(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}\right)$$

Remark 4.3.3. The closure of local conditions is redundant when $\mathbf{H}^1(K_{\mathfrak{u}},T)$ is finitely or cofinitely generated, but it is required in further generalizations of this theory (see, for example, in [LZ25]).

Remark 4.3.4. By Proposition 4.2.28, a Selmer structure depends only on the local conditions. If we set $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T) := \mathbf{H}^1_{\mathfrak{f}}(K_{\mathfrak{u}},T)$ whenever $\mathfrak{u} \notin \Sigma_{\mathcal{F}}$, then

$$\mathbf{H}^1_{\mathcal{F}}(K,T) = \ker\left(\mathbf{H}^1(K,T) \to \prod_{\mathfrak{u}\in\mathcal{U}(\mathbb{P}_K)} \frac{\mathbf{H}^1(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}\right)$$

Remark 4.3.5. Local conditions propagate to quotients and submodules as in Definitions 2.2.7 and 2.2.8.

We can recover the Selmer group as the limit of the propagated Selmer groups with finite coefficients.

Proposition 4.3.6. ([Swe22, Proposition 2.5.6]) Let T be a profinite abelian group and let $\mathcal{F}$ be a Selmer structure defined on T. Then

$$\mathbf{H}^1_{\mathcal{F}}(K,T) = \varprojlim_{T\twoheadrightarrow T'} \mathbf{H}^1_{\mathcal{F}}(K,T')$$

where the limit is taken over the finite quotients T' of T and the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K,T')$ is obtained from the propagated Selmer structure, as defined in Definition 2.2.7.

Proof. Since the inverse limit is left exact,

$$\begin{aligned}\varprojlim_{T\twoheadrightarrow T'} \mathbf{H}^1_{\mathcal{F}}(K,T) &= \varprojlim_{T\twoheadrightarrow T'} \ker\left(\mathbf{H}^1\big(K^{\Sigma_{\mathcal{F}}}/K,T'\big) \to \prod_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},T')\right)\\ &= \ker\left(\mathbf{H}^1\big(K^{\Sigma_{\mathcal{F}}}/K,T\big) \to \prod_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \varprojlim_{T\twoheadrightarrow T'} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},T')\right)\end{aligned}$$

By the definition of propagated Selmer structure and the closedness of $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)$, we obtain that

$$\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},T) = \varprojlim_{T\twoheadrightarrow T'} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},T')$$

which finishes the proof of this proposition. $\square$

The next dual proposition is proven similarly.

Proposition 4.3.7. ([Swe22, Proposition 2.5.6]) Let T be an indfinite group and let $\mathcal{F}$ be a Selmer structure defined on T. Then

$$\mathbf{H}^1_{\mathcal{F}}(K,T) = \varinjlim_{T' \hookrightarrow T} \mathbf{H}^1_{\mathcal{F}}(K,T')$$

where the limit is taken over the finite submodules T' of T and $\mathbf{H}^1_{\mathcal{F}}(K,T')$ is the Selmer group of the propagated Selmer structure, as defined in Definition 2.2.8.

The technical lemma below shows that patched Selmer groups can be obtained from patching classical Selmer groups whenever T is finite.

Lemma 4.3.8. Let $\mathcal{F} \leq \mathcal{G}$ be two Selmer structures defined on a finite Galois module T. Then there are sequences of classical Selmer structures $(\mathcal{F}_i)$ and $(\mathcal{G}_i)$ such that $\mathcal{F}_i \leq \mathcal{G}_i$ for all $i \in \mathbb{N}$ and for any representative $(\ell_i)_{i\in\mathbb{N}}$ of an ultraprime $\mathfrak{u}$,

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T) = \mathcal{U}\big(H^1_{\mathcal{F}_i}(K_{\ell_i},T)\big),\ \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T) = \mathcal{U}\big(H^1_{\mathcal{G}_i}(K_{\ell_i},T)\big)$$

Moreover, we can construct the patched Selmer group as the ultraproduct of classical Selmer groups

$$\mathbf{H}^1_{\mathcal{F}}(K,T) = \mathcal{U}\big(H^1_{\mathcal{F}_i}(K,T)\big),\ \mathbf{H}^1_{\mathcal{G}}(K,T) = \mathcal{U}\big(H^1_{\mathcal{G}_i}(K,T)\big)$$

Proof. For an ultraprime $\mathfrak{u} \in \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}$, we have fixed a representing sequence (ℓ_i). We can assume, without loss of generality, that no prime appears in two different sequences on the same entry. Let $W_{\mathfrak{u}}$ be the set of indices such that there is an isomorphism

$$\varphi_i^{\mathfrak{u}}:\ H^1(K_{\ell_i},T) \cong \mathbf{H}^1(K_{\mathfrak{u}},T)$$

Note that Corollary 4.2.10 implies that $W_{\mathfrak{u}} \in \mathcal{U}$.

For every $i \in \mathbb{N}$, define the classical Selmer structure by the set of primes

$$\Sigma_{\mathcal{F}_i} = \{\ell_i :\ \mathfrak{u} = (\ell_i)_{i\in\mathbb{N}} \in \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}\}$$

and local conditions

$$\begin{aligned} H^1_{\mathcal{F}_i}(K_{\ell_i},T) &= (\varphi_i^{\mathfrak{u}})^{-1}\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T) && \text{if } i \in W_{\mathfrak{u}} \\ H^1_{\mathcal{F}_i}(K_{\ell_i},T) &= 0 && \text{if } i \notin W_{\mathfrak{u}} \end{aligned}$$

This definition implies that $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) = \mathcal{U}_i\big(H^1_{\mathcal{F}}(K_{\ell_i}, T)\big)$. Therefore, Proposition 4.1.20 implies that

$$\ker\left[\mathbf{H}^1(K^{\mathfrak{n}}/K, T) \to \prod_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T)\right] =$$

$$\ker\left[\mathcal{U}\big(H^1(K^{n_i}/K, T)\big) \to \mathcal{U}\left(\prod_{\ell_i|n_i} H^1_{/\mathcal{F}}(K_{\ell_i}, T)\right)\right]$$

Again by the exactness of the ultraproduct given in Proposition 4.1.20 implies that

$$\mathbf{H}^1_{\mathcal{F}}(K, T) = \mathcal{U}\big(H^1_{\mathcal{F}_i}(K, T)\big)$$

Similarly, define Selmer structures $\mathcal{G}_i$ by the set of primes

$$\Sigma_{\mathcal{G}_i} = \{\ell_i :\ \mathfrak{u} = (\ell_i)_{i\in\mathbb{N}} \in \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}\}$$

and local conditions

$$\begin{array}{ll} H^1_{\mathcal{G}_i}(K_{\ell_i}, T) = (\varphi_i^{\mathfrak{u}})^{-1}\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) & \text{if } i \in W_{\mathfrak{u}} \\ H^1_{\mathcal{G}_i}(K_{\ell_i}, T) = H^1(K_{\mathfrak{u}_i}, T) & \text{if } i \notin W_{\mathfrak{u}} \end{array}$$

By construction $\mathcal{F}_i \leq \mathcal{G}_i$ and, analogously,

$$\mathbf{H}^1_{\mathcal{G}}(K, T) = \mathcal{U}\big(H^1_{\mathcal{G}_i}(K, T)\big) \qquad \square$$

4.3.2 Dual patched Selmer structures

In this section, we develop an extension of the theory in [Swe22] that generalises Proposition 2.2.6. In order to do this, we need to use dual Selmer structures, which were defined We can use local duality to define dual Selmer structures as in Definition 2.2.4

Definition 4.3.9 (Dual Selmer structure)**.** Let $\mathcal{F}$ be a Selmer structure on T. The we can define a *dual Selmer structure* $\mathcal{F}^*$ on T^* by defining the local condition $\mathbf{H}^1_{\mathcal{F}^*}(K_{\mathfrak{u}}, T^*)$ as the annihilator of $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ under the local duality pairing in Proposition 4.2.18

When T is a finite group, dual patched Selmer structures can be also obtained by patching the dual Selmer structures in the classical setting.

Lemma 4.3.10. Let $\mathcal{F}$ a patched Selmer structure defined on a finite group T and let $\mathcal{F}_i$ be classical Selmer structures such that

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) = \mathcal{U}\big(H^1_{\mathcal{F}_i}(K_{\ell_i}, T)\big)$$

for every ultraprime $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}}$. Then, we have that

$$\mathbf{H}^1_{\mathcal{F}^*}(K_{\mathfrak{u}}, T^*) = \mathcal{U}\big(H^1_{\mathcal{F}_i^*}(K_{\ell_i}, T^*)\big)$$

Proof. By Definition 4.3.9, for every ultraprime $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}}$, we have that

$$\mathbf{H}^1_{\mathcal{F}^*}(K_{\mathfrak{u}}, T^*) = \ker\left(H^1(K_{\mathfrak{u}}, T^*) \to \mathrm{Hom}\left(H^1_{\mathcal{F}}(K_{\mathfrak{u}}, T), \frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right)$$

where the map is induced by local duality, as stated in Proposition 4.2.18. This map can be also written as the composite

$$\begin{aligned}\mathcal{U}\big(H^1(K_{\ell_i}, T^*)\big) &\to \mathcal{U}\left(\mathrm{Hom}\left(H^1_{\mathcal{F}_i}(K_{\ell_i}, T), \frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right) \\ &\hookrightarrow \mathrm{Hom}\left(\mathcal{U}\big(H^1_{\mathcal{F}_i}(K_{\ell_i}, T)\big), \mathcal{U}\left(\frac{(\#T)^{-1}\mathbb{Z}}{\mathbb{Z}}\right)\right)\end{aligned}$$

where the second map is the injection given in Proposition 4.1.24. By Proposition 2.1.18 and 4.1.20, we can compute the kernel of this map as

$$\mathbf{H}^1_{\mathcal{F}^*}(K_{\mathfrak{u}}, T^*) = \mathcal{U}\big(H^1_{\mathcal{F}_i^*}(K_{\ell_i}, T^*)\big) \qquad \square$$

This construction can be used to obtain a global duality exact sequence in the patched setting.

Proposition 4.3.11. (Global duality) Let $\mathcal{F} \leq \mathcal{G}$ be two Selmer structures. Then there is a global duality exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K, T) \arrow[r] & \mathbf{H}^1_{\mathcal{G}}(K, T) \arrow[r] & \displaystyle\bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}}\cup\Sigma_{\mathcal{G}}} \frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)} \arrow[dll] \\ & \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \arrow[r] & \mathbf{H}^1_{\mathcal{G}^*}(K, T^*)^\vee \arrow[r] & 0 \end{tikzcd}$$

Proof. Assume first that T is finite. Let $\mathfrak{n} = (n_k)_{k\in\mathbb{N}}$ be the square-free product of all ultraprimes in $\Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}$. By Lemma 4.3.8, there are sequences of classical Selmer structures $(\mathcal{F}_i)$ and $(\mathcal{G}_i)$, where $\mathcal{F}_i \leq \mathcal{G}_i$, and

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) = \mathcal{U}_i\big(H^1_{\mathcal{F}_i}(K_{\ell_i}, T)\big),\ \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) = \mathcal{U}_i\big(H^1_{\mathcal{G}_i}(K_{\ell_i}, T)\big)$$

In addition, Lemma 4.3.8 also gives that

$$\mathbf{H}^1_{\mathcal{F}}(K, T) = \mathcal{U}_i\big(H^1_{\mathcal{F}_i}(K, T)\big),\ \mathbf{H}^1_{\mathcal{G}}(K, T) = \mathcal{U}_i\big(H^1_{\mathcal{G}_i}(K, T)\big)$$

By Lemma 4.3.10 and Proposition 4.1.20, the dual Selmer groups can be also obtained as an ultraproduct of classical Selmer groups:

$$\begin{aligned}\mathbf{H}^1_{\mathcal{F}^*}(K,T^*) &= \mathcal{U}_i\big(H^1_{(\mathcal{F}_i)^*}(K,T^*)\big)\\ \mathbf{H}^1_{\mathcal{G}^*}(K,T^*) &= \mathcal{U}_i\big(H^1_{(\mathcal{G}_i)^*}(K,T^*)\big)\end{aligned}$$

Then the global duality exact sequence is

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U}\big(H^1_{\mathcal{F}_i}(K,T)\big) \arrow[r] & \mathcal{U}\big(H^1_{\mathcal{G}_i}(K,T)\big) \arrow[r] & \mathcal{U}\left(\bigoplus_{\ell_i \mid n_i} \dfrac{H^1_{\mathcal{G}_i}(K_{\ell_i},T)}{H^1_{\mathcal{F}_i}(K_{\ell_i},T)}\right) \arrow[dll] \\ & \mathcal{U}\big(H^1_{\mathcal{F}^*}(K,T^*)^\vee\big) \arrow[r] & \mathcal{U}\big(H^1_{\mathcal{G}^*}(K,T^*)^\vee\big) \arrow[r] & 0 \end{tikzcd}$$

which is exact by Proposition 4.1.20.

When T is profinite (resp. ind-finite), then Proposition 4.3.6 (resp. Proposition 4.3.7) implies that the above exact sequence can be obtained as the inverse (resp. direct) limit of the associated exact sequences for the finite quotients (resp. submodules) of T. The proof is thus concluded since the inverse limit of finite groups (resp. direct limit) is an exact functor. $\square$

Chapter 5

Ultra Kolyvagin systems

This chapter builds on Chapter 4 to construct a theory of ultra Kolyvagin systems. They can be constructed naturally in the setting of patched cohomology and compared with limits of classical Kolyvagin systems.

The second half of this chapter (§5.3 and §5.4) concerns the application of this setting to the case in which R is a discrete valuation ring. Particularly, we generalise Theorems 3.2.3, 3.3.7 and 3.4.1. In this setting, Theorem 3.3.7 has a nicer reinterpretation since it determines the Selmer group up to isomorphism.

Arguably, the formalism of ultra Kolyvagin systems do not suppose any advantage in the study of Selmer groups whose coefficient ring R is a discrete valuation ring. The reason is that ultra Kolyvagin systems are limits of classical Kolyvagin systems, which completely determined the Fitting ideals of the Selmer group and, hence, its isomorphism class.

The main advantage of the use of ultra Kolyvagin systems will become apparent when R is the Iwasawa algebra (see Chapter 6). In this situation, the theta ideals of the classical Kolyvagin systems could not be linked, using the classical theory, with the Fitting ideals of the Selmer group, but that is possible the formalism of ultra Kolyvagin systems, in combination with the structure theorem of finitely generated Iwasawa modules.

5.1 Patched Selmer modules

The first part of this chapter only assumes a weak assumption on the ring R. However, we will later assume that R is either a discrete valuation ring or the Iwasawa algebra, in order to construct the rank reduction map between biduals of Selmer groups.

Assumption (R-lim). Let R be a local, complete, noetherian domain with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Assume further that R can be written as an inverse limit

$$R = \varprojlim_{k \in \mathbb{N}} R_k$$

where R_k are self-injective quotients of R and $R_1 = R/\mathfrak{m}$. Denote by I_k the ideal of R inducing the quotient $R_k = R/I_k$.

Remark 5.1.1. By Proposition B.1.13, for every $m < n$, the ideal $R_n[I_m]$ is a principal ideal of R_n.

In addition, let T be a $R[[G_K]]$-module that is finitely generated as R-module. Similarly, we denote $T_k = T \otimes_R R_k$. It is important to note that, since R is complete and T is finitely generated, then T_k is a cofinal sequence, so

$$T = \varprojlim_{n \in \mathbb{N}} T_k$$

Remark 5.1.2. Discrete valuation rings and the Iwasawa algebra satisfy Assumption (R-lim). Indeed, for discrete valuation rings, the natural choice $I_k = \mathfrak{m}^k$ satisfies all the hypotheses.

When R is the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[X]]$, there are two natural choices for the ideals I_k, both satisfying all the hypothesis:

$$I_k = (p^k, T^k), \ I'_k = (p^k, (T+1)^{p^k} - 1)$$

We will more frequently use the choice I_k although, in some specific context, it is more convenient to use the ideals I'_k.

The advantage of patched cohomology is that we can develop a theory of Kolyvagin systems for the rings satisfying Assumption (R-lim). That was not possible in the classical setting, since the finiteness of the ring R was a requirement for constructing Kolyvagin primes using the Chebotarev density theorem. We keep assuming Assumptions (BI), restated below.

Assumption (BI). In addition, we assume the following assumptions:

- (T1) $T/\mathfrak{m}T$ is an irreducible $k[[G_K]]$-module.
- (T2) There exists $\tau \in G_{K_M}$ such that $T/(\tau - 1)T \cong R$ as R-modules.

- (T3) $H^1(K(T)_{p^\infty}/K, T) = H^1(K(T)_{p^\infty}/K, T^*(1)) = 0$.

Remark 5.1.3. Note that Proposition 2.1.6 implies that $H^0(K, T')$ vanishes for every quotient T' of T. For all square-free products of unramified ultraprimes at T, then $\mathbf{H}^0(K^{\mathfrak{n}}/K, T')$ vanishes, which also implies that $\mathbf{H}^0(K, T') = 0$. Similarly, $\mathbf{H}^0(K, T'') = 0$ for all submodules T'' of T^*.

We can use this to show that, under these assumptions, profinite Selmer groups are torsion free.

Proposition 5.1.4. Let T be a Galois representation satisfying Assumptions (R-lim) and (BI). For every Selmer structure $\mathcal{F}$, the Selmer group

$$\mathbf{H}^1_{\mathcal{F}}(K, T)$$

is a torsion-free R-module.

Proof. Let a in R and consider the exact sequence

$$0 \longrightarrow T \longrightarrow T \longrightarrow T/aT \longrightarrow 0$$

By Remark 5.1.3 and Proposition 4.3.11, we have that

$$\mathbf{H}^1(K^{\Sigma_{\mathcal{F}}}/K, T)[a] = 0$$

Since that holds for every $a \in R$ and Definition 4.3.1 defines the Selmer group as a subgroup of the global patched cohomology group, we can deduce that

$$\mathbf{H}^1_{\mathcal{F}}(K, T)_{\mathrm{tors}} = 0$$

□

5.1.1 Cartesian Selmer structures and core rank

When the coefficient ring of a Galois representation is an integral domain, the cartesian patched Selmer structures are defined below.

Definition 5.1.5 (Cartesian local condition)**.** A local condition $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ is called *cartesian* if the quotient

$$\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T) = \frac{\mathbf{H}^1(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)}$$

is a torsion-free R-module (see Notation 5.1.6). A Selmer structure is said to be cartesian if all of its local conditions are cartesian.

Notation 5.1.6. We clarify the definition of torsion element: some element $a \in M$ of an R-module M is said to be *torsion* when $\mathrm{Ann}_R(a) \neq 0$.

Remark 5.1.7. The above definition might seem unrelated to the previous definition of cartesian local condition given in Definition 2.2.11. However, it is shown in [MR04, Lemma 3.7.1] that, when R is a discrete valuation ring, if a Selmer structure on T is cartesian as in Definition 5.1.5, the propagated Selmer structure to a finite quotient $T/\mathfrak{m}^i$ will be cartesian in the sense of Definition 2.2.11.

Remark 5.1.8. In order to check that a Selmer structure $\mathcal{F}$ is cartesian, it is enough to see that local conditions are cartesians for ultraprimes $\mathfrak{u} \in \Sigma_{\mathcal{F}}$. Indeed, finite cohomology groups are always cartesian local conditions since

$$\mathbf{H}^1_s(K_{\mathfrak{u}}, T) \subset \mathrm{Hom}(\mathcal{I}_{\mathfrak{u}}, T)$$

is R-torsion-free since R is a domain and T is a free module.

Remark 5.1.9. Considering the computations in §6.1.3 when R is the Iwasawa algebra, it might seem reasonable to define cartesian conditions imposing that $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ is a free R-module. However, that definition presents some technical issues since we cannot guarantee that $\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T)$ is a free module in general.

The main advantage of cartesian structures is that they allow the study of Selmer groups of T by projecting to the cohomology of T_1. In order to do that, we assume the following mild assumption.

Assumption (Proj). There is a square-free product of ultraprimes $\mathfrak{n}$, divisible by all the primes in $\Sigma_{\mathcal{F}}$, such that the following projection map is injective:

$$\mathbf{H}^1(K^{\mathfrak{n}}, T) \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K, T_1)$$

Remark 5.1.10. When Assumptions (BI) hold and R is a principal ring, then Assumption (Proj) also holds for every square-free product of ultraprimes $\mathfrak{n}$. Indeed, by Remark 5.1.3, we have that $\mathbf{H}^0(K^{\mathfrak{n}}/K, T)$ for every square-free product of ultraprimes $\mathfrak{n}$. Then the short exact sequence

$$0 \longrightarrow T \longrightarrow T \longrightarrow T_1 \longrightarrow 0$$

induces, by Corollary 4.2.24, an injection

$$\mathbf{H}^1(K^{\mathfrak{n}}/K, T) \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K, T_1)$$

Remark 5.1.11. When R is the Iwasawa algebra, it will be shown in Proposition 6.1.2 that Assumption (Proj) is equivalent to the lack of finite submodules in the local cohomology groups.

Proposition 5.1.12. Let $\mathcal{F}$ be a cartesian Selmer structure. If $\mathfrak{n}$ is a square-free product of all the ultraprimes in $\Sigma_{\mathcal{F}}$, for every ideal I of R, there is an injection

$$\mathbf{H}^1_{\mathcal{F}}(K,T)/I \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K,T)/I \tag{5.1}$$

Proof. By Definition 4.3.1 and Remark 4.3.4, there is an injection

$$\frac{\mathbf{H}^1(K^{\mathfrak{n}}/K,T)}{\mathbf{H}^1_{\mathcal{F}}(K,T)} \hookrightarrow \prod_{\mathfrak{u}|\mathfrak{n}} \frac{\mathbf{H}^1(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}$$

By Definition 5.1.5,

$$\left(\frac{\mathbf{H}^1(K^{\mathfrak{n}}/K,T)}{\mathbf{H}^1_{\mathcal{F}}(K,T)}\right)[I] = 0 \tag{5.2}$$

Then the proof is completed by Lemma 5.1.13 below. □

Lemma 5.1.13. Let $\mu : A \hookrightarrow B$ be an injection of R-modules such that the quotient $B/\mu(A)$ is torsion free. Then, for every ideal $I \subset R$, there is an injection

$$A/I \hookrightarrow B/I$$

Proof. Assume for the sake of contradiction that the map $A/I \to B/I$ is not injective. Then there exists some $a \in A$ such that

$$a \notin IA, \quad \mu(a) \in IB$$

Then there exists some $b \in B$ and $\lambda \in I$ such that $\mu(a) = \lambda b$. Since a is not I-divisible in A, then $b \notin \mu(A)$. Hence $b + \mu(A)$ induces a non-trivial I-torsion class in the quotient $B/\mu(A)$. By assumption, $b \in \mu(A)$, which leads to a contradiction. □

Corollary 5.1.14. If Assumption (Proj) holds and $\mathcal{F}$ is a cartesian Selmer structure, the following projection map is injective.

$$\mathbf{H}^1_{\mathcal{F}}(K,T)/\mathfrak{m} \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T_1)$$

Proof. Let $\mathfrak{n}$ be the square-free product of all the ultraprimes in $\Sigma_{\mathcal{F}}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{H}^1_{\mathcal{F}}(K,T)/\mathfrak{m} \arrow[r] \arrow[d, tail] & \mathbf{H}^1_{\mathcal{F}}(K,T_1) \arrow[d] \\
\mathbf{H}^1(K^{\mathfrak{n}}/K,T)/\mathfrak{m} \arrow[r, tail] & \mathbf{H}^1(K^{\mathfrak{n}}/K,T_1)
\end{tikzcd}$$

In this diagram, the left vertical map is injective by Proposition 5.1.12 and the bottom horizontal map is injective by Assumption (Proj). Hence the top horizontal map needs to be injective. $\square$

The previous corollary implies that the analogous maps induced by the quotients R_k are also injective.

Proposition 5.1.15. If Assumptions (R-lim) and (Proj) hold and $\mathcal{F}$ is a cartesian Selmer structure, the map

$$\mathbf{H}^1_{\mathcal{F}}(K,T) \otimes_R R_k \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T_k)$$

is injective for all $k \in \mathbb{N}$.

Proof. Recall that there are ideals I_k of R such that $R_k = R/I_k$. Let

$$x \in \ker\Big(\mathbf{H}^1_{\mathcal{F}}(K,T) \to \mathbf{H}^1_{\mathcal{F}}(K,T_k)\Big)$$

Then, x is also in the kernel of the projection map to $\mathbf{H}^1_{\mathcal{F}}(K,T_1)$, so Corollary 5.1.14 implies the existence of an element $a \in \mathfrak{m}$ and $y \in \mathbf{H}^1_{\mathcal{F}}(K,T)$ such that $x = ay$.

We repeat this process with the class y until we find an element that is not in the kernel of the projection map to $\mathbf{H}^1_{\mathcal{F}}(K,T_1)$, so we construct an element $z \in \mathbf{H}^1_{\mathcal{F}}(K,T)$ such that $x = bz$ for some $b \in R$. We aim to show that $b \in I_k$. Note that, in case that the condition on z is never achieved, we will eventually find an element $b \in I_k$, which would prove the proposition.

Assume by contradiction that it is not and let $\overline{b}$ be the projection in R_k. By assumption,

$$\Pi(z) \in \mathbf{H}^1_{\mathcal{F}}(K,T_k)[\overline{b}]$$

where Π represents the canonical projection map. However, since $\overline{b} \neq 0$ divides any element in $R_k[\mathfrak{m}]$, Lemma 5.1.16 below implies that the projection of z to the Selmer group of T_1 vanishes, which leads to a contradiction. $\square$

Lemma 5.1.16. Let $I = R_k[\mathfrak{m}]$. Then the following holds:

$$\ker\big(\mathbf{H}^1_{\mathcal{F}}(K, T_k) \to \mathbf{H}^1_{\mathcal{F}}(K, T_1)\big) = \mathbf{H}^1_{\mathcal{F}}(K, T_k)[I]$$

where the map is induced by the projection map $T_k \twoheadrightarrow T_1$.

Proof. Since R_k is self-injective, I is the minimal non-zero ideal of R_k, so it is principal. Choose a generator $x \in I$. Since T_k is a free R_k-module, there is a short exact sequence

$$0 \longrightarrow T_1 \xrightarrow{[x]} T_k \longrightarrow T_k/T_1 \longrightarrow 0$$

Since T_k/T_1 is a quotient of T, Remark 5.1.3 and Corollary 4.2.24 induces an injection

$$[x]: \ \mathbf{H}^1_{\mathcal{F}}(K, T_1) \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$

Note that the following composition is the multiplication by x:

$$\mathbf{H}^1_{\mathcal{F}}(K, T_k) \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K, T_1) \stackrel{[x]}{\rightarrowtail} \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$

Since the second map is injective, then

$$\ker\big(\mathbf{H}^1_{\mathcal{F}}(K, T_k) \to \mathbf{H}^1_{\mathcal{F}}(K, T_1)\big) = \mathbf{H}^1_{\mathcal{F}}(K, T_k)[x] \qquad \square$$

Now we adapt Proposition 2.2.10 to the patched cohomology setting.

Proposition 5.1.17. Under Assumptions (BI), for every finitely generated ideal I of R, the inclusion $T^*[I] \hookrightarrow T^*$ induces an isomorphism

$$H^1_{\mathcal{F}^*}(K, T^*[I]) \cong H^1_{\mathcal{F}^*}(K, T^*)[I]$$

Proof. Consider the following short exact sequence

$$0 \longrightarrow T^*[I] \longrightarrow T^* \longrightarrow T^*/T^*[I] \longrightarrow 0$$

By Corollary 4.2.24 and Remark 5.1.3, there is another exact sequence

$$\mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*[I]\big) \rightarrowtail \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big) \longrightarrow \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*/T^*[I]\big) \tag{5.3}$$

Let $\{r_1, \cdots, r_d\}$ be a minimal set of generators of I. Then there is an injection

$$T^*/T^*[I] \hookrightarrow (T^*)^d, \ t \mapsto (r_1 t, \dots, r_d t)$$

This injection fits into the exact sequence

$$0 \longrightarrow T^*/T^*[I] \longrightarrow (T^*)^d \longrightarrow T^*/(r_1) \times \cdots \times T^*/(r_d)$$

By Proposition 4.1.23, the patched cohomology commutes with the direct product, so Remark 5.1.3 implies that

$$\mathbf{H}^0\big(K^{\mathfrak{n}}/K, T^*/(r_1) \times \cdots \times T^*/(r_d)\big) = 0$$

Hence,

$$\mathbf{H}^0\Big(\operatorname{coker}\big(T^*/T^*[I] \hookrightarrow (T^*)^d\big)\Big) = 0$$

Then Corollary 4.2.24 induces an injection

$$\mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*/T^*[I]\big) \hookrightarrow \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big)^d$$

Since the composition

$$\mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big) \to \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*/T^*[I]\big) \hookrightarrow \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big)^d$$

is the multiplication by $(r_1, \ldots, r_d)$, its kernel coincides with the torsion subgroup $\mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big)[I]$. Since the second map is injective, the kernel of the composition coincides with the kernel of the first map. From (5.3), we can conclude that

$$\mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*[I]\big) = \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big)[I]$$

Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[I]) \arrow[r, tail] \arrow[d] & \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*[I]\big) \arrow[r] \arrow[d, "\sim"] & \bigoplus_{\mathfrak{u} \in \Sigma_{\mathcal{F}}} \mathbf{H}^1_{/\mathcal{F}^*}(K_{\mathfrak{u}}, T^*[I]) \arrow[d] \\
\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)[I] \arrow[r, tail] & \mathbf{H}^1\big(K^{\mathfrak{n}}/K, T^*\big)[I] \arrow[r] & \bigoplus_{\mathfrak{u} \in \Sigma_{\mathcal{F}}} \mathbf{H}^1_{/\mathcal{F}^*}(K_{\mathfrak{u}}, T^*)
\end{tikzcd}$$

We have already seen that the middle vertical arrow is an isomorphism. The right vertical arrow is injective, since its dual

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \twoheadrightarrow \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/I)$$

is surjective by definition of propagated Selmer structures. Therefore, the leftmost vertical arrow is also an isomorphism. $\square$

We can now define the concept of core rank. We give a definition that applies for any Selmer structure $\mathcal{F}$, not necessarily cartesian.

Definition 5.1.18. Let $\mathcal{F}$ be a Selmer structure. We define the *core rank* of $\mathcal{F}$ as

$$\chi(\mathcal{F}) = \mathrm{rank}_R\ \mathbf{H}^1_{\mathcal{F}}(K, T) - \mathrm{rank}_R\ \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee$$

Notation 5.1.19. Similarly, we clarify our understanding of the rank of an R-module M. Since R is an integral domain, the ideal (0) is prime, so we can define the rank of M as the dimension of $M_{(0)}$ as an $R_{(0)}$-vector space. Note that, since the localisation is an exact functor, the rank is an additive function.

Remark 5.1.20. The core rank is well defined since both $\mathbf{H}^1_{\mathcal{F}}(K, T)$ and $\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee$ are finitely generated modules.

5.1.2 Kolyvagin ultraprimes

In the patched cohomology setting, we can define Kolyvagin ultraprimes similarly. Recall that Assumption (BI) implies the existence of $\tau \in G_{K_{p^\infty}}$ such that $T/(\tau - 1)$ is a free R-module of rank one.

Definition 5.1.21. An ultraprime is said to be a *Kolyvagin ultraprime* if $\mathrm{Frob}_{\mathfrak{u}}$ is conjugate to τ in $\mathrm{Gal}(K(T)_{p^\infty}/K)$.

Notation 5.1.22. Similarly, we denote by $\mathcal{P}$ the set of Kolyvagin ultraprimes, and we denote by $\mathcal{N}$ and $\mathcal{N}_i$ the set of square-free products of Kolyvagin primes and exactly i Kolyvagin primes, respectively.

Notation 5.1.23. For $\mathfrak{n} \in \mathcal{N}$, we denote by $\nu(\mathfrak{n})$ the number of primes dividing $\mathfrak{n}$.

Remark 5.1.24. Proposition 4.1.32 guarantees the existence of Kolyvagin ultraprimes. Note that $K(T)_{p^\infty}/K$ might be an infinite extension, so we cannot apply the Chebotarev density theorem to guarantee the existence of classical Kolyvagin primes.

We can describe explicitly the local cohomology of Kolyvagin ultraprimes.

Proposition 5.1.25. Let $\mathfrak{u}$ be a Kolyvagin ultraprime. Then there is a homomorphism

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \cong T/(\mathrm{Frob}_{\mathfrak{u}} - 1)T \cong T/(\tau - 1)T \cong R$$

Proof. Recall that $T_k = T \otimes_R R_k$ and let $(\ell_i)_{i\in\mathbb{N}}$ be the Kolyvagin ultraprime. For every $k \in \mathbb{N}$, denote by $\alpha_n \in \mathbb{N}$ the minimal natural number such that $p^{\alpha_n} R_k = 0$.

xThen for $\mathcal{U}$-many i, the Frobenius element Frob_{ℓ_i} is conjugate to τ in $\mathrm{Gal}(K(T_k)_{p^{\alpha_n}}/K)$. Therefore, ℓ_i is a Kolyvagin prime for $\mathcal{U}$-many i. For those i, we have that

$$H^1_{\mathrm{f}}(K_{\ell_i}, T_k) \cong \frac{T_k}{(\mathrm{Frob}_{\ell_i} - 1)T_k} \cong \frac{T_k}{(\tau - 1)T_k} \cong R_k$$

By Remark 4.2.16,

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T_k) = \mathcal{U}_i\big(H^1_{\mathrm{f}}(K_{\ell_i}, T_k)\big) \cong \frac{T_k}{(\mathrm{Frob}_{\mathfrak{u}} - 1)T_k} \cong \frac{T_k}{(\tau - 1)T_k} \cong R_k$$

Taking limits, we get that

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \cong \frac{T}{(\mathrm{Frob}_{\mathfrak{u}} - 1)T} \cong \frac{T}{(\tau - 1)T} \cong R \qquad \square$$

We can use a similar argument for constructing the patched transverse local conditions.

Definition 5.1.26. Let $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}} \in \mathcal{P}$ be a Kolyvagin ultraprime. For every $i \in N$, recall that $K(\ell_i)$ is the maximal p-extension inside the ray class field modulo ℓ_i. We define the transverse local condition as

$$\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T) = \varprojlim_{n\in\mathbb{N}} \mathcal{U}_i\Big(\mathrm{Im}\left(\inf :\ H^1(K(\ell_i)_{\ell_i}/K, T) \to H^1(K_{\ell_i}, T)\right)\Big)$$

We can now prove the basic properties of the transverse cohomology used to develop the theory of Kolyvagin systems.

Proposition 5.1.27. Let $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}} \in \mathcal{P}$ be a Kolyvagin ultraprime. Then the transverse cohomology group is a free, cyclic R-module and the local patched cohomology splits as

$$\mathbf{H}^1(K_{\mathfrak{u}}, T) = \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \oplus \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T)$$

Proof. For every $k \in \mathbb{N}$, denote by S_k the set of indices such that $\ell_i \in \mathcal{P}(T_k)$. Note that $S_k \in \mathcal{U}$ for all $k \in \mathbb{N}$. For $i \in S_k$, [MR04, Lemmas 1.2.1 and 1.2.4] imply that

$$\mathbf{H}^1_{\mathrm{tr}}(K_{\ell_i}, T_k) \cong R_k, \quad \mathbf{H}^1(K_{\ell_i}, T_k) \cong \mathbf{H}^1_{\mathrm{f}}(K_{\ell_i}, T_k) \oplus \mathbf{H}^1_{\mathrm{tr}}(K_{\ell_i}, T_k)$$

By Lemma 4.1.21, the finite patched version also satisfies that

$$\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T_k) = \mathcal{U}_i\big(H^1_{\mathrm{tr}}(K_{\ell_i}, T_k)\big) \cong R_k \tag{5.4}$$

Taking limits, we obtain that

$$\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T) = \varprojlim_{k\in\mathbb{N}} \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T_k) \cong \varprojlim_{n\in\mathbb{N}} R_k = R$$

It is worth mentioning that, since the isomorphisms in (5.4) are not canonical, we cannot guarantee that the transition maps in both limits coincide. However, it is possible to guarantee that the transition maps are surjective, with implies that $\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T)$ is a free module of rank one.

Proposition 4.1.23 implies that

$$\mathbf{H}^1(K_{\mathfrak{u}}, T_k) \cong \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T_k) \oplus \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T_k)$$

And, taking limits,

$$\mathbf{H}^1(K_{\mathfrak{u}}, T) \cong \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \oplus \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T) \qquad \square$$

Remark 5.1.28. Note that Proposition 5.1.27 gives a canonical isomorphism

$$\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T) \cong \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T)$$

We can also construct a finite-singular map in the patched setting. In order to do that, we need to define a generalisation of the Galois groups $\mathcal{G}_\ell$ for ultraprimes.

Definition 5.1.29. Let $\mathfrak{u} = (\ell_i)_{i\in\mathbb{N}} \in \mathcal{U}(\mathbb{P}_K)$ be an ultraprime. Define the group

$$\mathcal{G}_{\mathfrak{u}} := \varprojlim_{k\in\mathbb{N}} \mathcal{U}_i\big(\mathcal{G}_{\ell_i}/p^k\big)$$

Remark 5.1.30. Note that Definition 5.1.29 is independent of the sequence representing the ultraprime. In addition, if $\mathfrak{u}$ is the constant ultraprime (ℓ), then $\mathcal{G}_{\mathfrak{u}}$ coincides with the group $\mathcal{G}_\ell$.

Remark 5.1.31. If $\mathfrak{u}$ is a Kolyvagin ultraprime, then $G_{\mathfrak{u}}$ is isomorphic to $\mathbb{Z}_p$.

Proposition 5.1.32. There is a canonical isomorphism

$$\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \cong \mathrm{Hom}(\mathcal{I}_{\mathfrak{u}}, T)^{\mathrm{Frob}_{\mathfrak{u}}=1}$$

Proof. This is obtained from the inflation-restriction sequence, in the identifications of Propositions 4.2.9 and 4.2.12 and the cohomological dimension of $\langle \mathrm{Frob}_{\mathfrak{u}} \rangle$ being one. $\square$

We can restate the isomorphism in Proposition 5.1.32 in the following way:

Corollary 5.1.33. If $\mathfrak{u}$ is a Kolyvagin ultraprime, there is a canonical isomorphism

$$\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \otimes_{\mathbb{Z}_p} \mathcal{G}_{\mathfrak{u}} \cong T^{\mathrm{Frob}_{\mathfrak{u}}=1}$$

Proof. By Definition 4.1.33, there is an isomorphism

$$\mathrm{Hom}(I_{\mathfrak{u}}, T) \to \mathrm{Hom}(\mathbb{Z}_p(1), T)$$

Since $\mathfrak{u} = (\ell_i)_{i \in \mathbb{N}}$ is a Kolyvagin ultraprime, then there is a canonical isomorphism $\mathcal{G}_{\ell_i}/p^k = \mu_{p^k}$ for $\mathcal{U}$-many i and all $k \in \mathbb{N}$. Those isomorphisms can be patched to give a canonical identification $\mathcal{G}_{\mathfrak{u}} = \mu_{p^\infty}$. Moreover, the fact that $\mathfrak{u}$ is a Kolyvagin prime implies that $\mathrm{Frob}_{\mathfrak{u}}$ acts trivially on the p-adic roots of unit, so $\mathbb{Z}_p$ and $\mathbb{Z}_p(1)$ are the same as $\langle \mathrm{Frob}_{\mathfrak{u}} \rangle$-modules. By Proposition 5.1.32

$$\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \otimes \mathcal{G}_{\mathfrak{u}} \cong \mathrm{Hom}(\mathbb{Z}_p(1), T)^{\mathrm{Frob}_{\mathfrak{u}}=1} \otimes \mathcal{G}_{\mathfrak{u}} = T^{\mathrm{Frob}_{\mathfrak{u}}=1} \qquad \square$$

Finite-singular maps also appear in this setting.

Proposition 5.1.34. There is a canonical isomorphism, known as *finite-singular map*,

$$\phi^{\mathrm{fs}}_{\mathfrak{u}} : \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \to \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \otimes_{\mathbb{Z}_p} \mathcal{G}_{\mathfrak{u}}$$

Proof. Let $P_{\mathfrak{u}}$ be the characteristic polynomial of the Galois action of $\mathrm{Frob}_{\mathfrak{u}}$ on T. The fact that $T/(\mathrm{Frob}_{\mathfrak{u}} - 1)$ is one dimensional, implies that $P_{\mathfrak{u}}(X) = (X-1)Q_{\mathfrak{u}}(X)$, where $Q_{\mathfrak{u}}$ is a polynomial satisfying that $Q_{\mathfrak{u}}(1) \neq 0$. Note that, since $P_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}})$ acts trivially on T, then $Q_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}})$ induces a map

$$Q_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}}) : T/(\mathrm{Frob}_{\mathfrak{u}} - 1)T \to T^{\mathrm{Frob}_{\mathfrak{u}}=1}$$

Consider then the composition,

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \longrightarrow T/(\mathrm{Frob}_{\mathfrak{u}} - 1)T \longrightarrow T^{\mathrm{Frob}_{\mathfrak{u}}=1} \longrightarrow \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \otimes \mathcal{G}_{\mathfrak{u}}$$

The first and third maps are isomorphisms by Proposition 5.1.25 and Corollary 5.1.33. Hence it only remains to see that $Q_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}})$ induces an isomorphism. Since $T/\mathfrak{m}$ is an $R/\mathfrak{m}$-vector space, the following map is an isomorphism:

$$\overline{Q_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}})}:\ T/(\mathrm{Frob}_{\mathfrak{u}}-1,\mathfrak{m})T \to (T/\mathfrak{m})^{\mathrm{Frob}_{\mathfrak{u}}=1}$$

Nakayama's lemma then implies that $Q_{\mathfrak{u}}(\mathrm{Frob}_{\mathfrak{u}})$ is surjective. Since $T/(\mathrm{Frob}_{\mathfrak{u}}-1)T$ and $T^{\mathrm{Frob}_{\mathfrak{u}}=1}$ are free R-modules of the same rank, then it is in fact an isomorphism. □

Remark 5.1.35. Note that, after choosing a generator of $\mathcal{G}_{\mathfrak{u}}$, the finite-singular map can be written as a well-determined map

$$\phi_{\mathfrak{u}}^{\mathrm{fs}}:\ \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},T) \to \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},T)$$

Note that, when $\mathfrak{u}$ is represented by the sequence $(\ell_i)_{i\in\mathbb{N}}$, then choosing a generator of $\mathcal{G}_{\mathfrak{u}}$ is the same as choosing a generator of $\mathcal{G}_{\ell_i}$ for $\mathcal{U}$-many i. In this case, the finite-singular maps $\phi_{\mathfrak{u}}^{\mathrm{fs}}$ can be obtained by patching $\phi_{\ell_i}^{\mathrm{fs}}$ in a natural way.

We can now use these local conditions to modify Selmer structures, similarly to Definition 2.2.17

Definition 5.1.36. Let $\mathcal{F}$ be a Selmer structure and let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be pairwise coprime square-free integers. Assume $\mathfrak{c}\in\mathcal{N}$. Define the Selmer structure $\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{c})$ by the local conditions

$$\mathbf{H}^1_{\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{u}},T) = \begin{cases} 0 & \text{if } \mathfrak{u}\mid\mathfrak{a} \\ \mathbf{H}^1(K_{\mathfrak{u}},T) & \text{if } \mathfrak{u}\mid\mathfrak{b} \\ \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}},T) & \text{if } \mathfrak{u}\mid\mathfrak{c} \\ \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T) & \text{otherwise} \end{cases}$$

We can show that modified Selmer structures are cartesian and compute their core rank.

Proposition 5.1.37. ([Sak18, Corollary 3.21]) Let $\mathcal{F}$ be a cartesian Selmer structure and let $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathcal{N}$ be pairwise coprime. Then $\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{c})$ is also cartesian and

$$\chi(\mathcal{F}_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{c})) = \chi(\mathcal{F}) + \nu(\mathfrak{b}) - \nu(\mathfrak{a})$$

Proof. For every Kolyvagin prime $\mathfrak{u} \in \mathcal{P}$, both $\mathbf{H}^1(K_{\mathfrak{u}}, T)$ and $\mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T)$ are torsion-free, so $\mathcal{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{c})$ is cartesian.

Proposition 4.3.11 produces an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K, T) \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{bc}}}(K, T) \arrow[r] & \oplus_{\mathfrak{u}|\mathfrak{bc}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T) \arrow[dll] \\
& \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^{\vee} \arrow[r] & \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{bc}}}(K, T^*)^{\vee} \arrow[r] & 0
\end{tikzcd}$$

Since the rank is additive, then $\chi(\mathcal{F}^{\mathfrak{bc}}) = \chi(\mathcal{F}) + \nu(\mathfrak{b}) + \nu(\mathfrak{c})$. Proposition 4.3.11 produces another exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{c})}(K, T) \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{bc}}}(K, T) \arrow[r] & \oplus_{\mathfrak{u}|\mathfrak{ac}} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T) \arrow[dll] \\
& \mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K, T^*)^{\vee} \arrow[r] & \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{bc}}}(K, T^*)^{\vee} \arrow[r] & 0
\end{tikzcd}$$

Thus,

$$\chi(\mathcal{F}^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{c})) = \chi(\mathcal{F}^{\mathfrak{bc}}) - \nu(\mathfrak{a}) - \nu(\mathfrak{c}) = \chi(\mathcal{F}) + \nu(\mathfrak{b}) - \nu(\mathfrak{a})$$

□

The analogue of Lemma 2.2.22 is a central technical result when working with Kolyvagin systems.

Lemma 5.1.38. Let $\mathcal{F}$ be a Selmer structure and let $\mathfrak{u} \in \mathcal{P}$ be an ultraprime satisfying that

$$\mathrm{loc}_{\mathfrak{u}} : \ \mathbf{H}^1_{\mathcal{F}}(K, T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$$

is surjective. Then $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K, T^*) = H^1_{(\mathcal{F}^*)_{\mathfrak{u}}}(K, T^*)$.

Proof. The surjectivity of $\mathrm{loc}_{\mathfrak{u}}$, together with the exact sequence of Proposition 4.3.11 with Selmer structures $\mathcal{F}_{\mathfrak{u}}$ and $\mathcal{F}$ implies that

$$\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}}(K, T^*) = \mathbf{H}^1_{\mathcal{F}^*}(K, T^*) \tag{5.5}$$

By construction, we have that

$$\mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}}}(K, T^*) = \mathbf{H}^1_{\mathcal{F}^*}(K, T^*) \cap \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K, T^*)$$

By (5.5)

$$\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{u}}(K,T^*) = \mathbf{H}^1_{(\mathcal{F}^*)^\mathfrak{u}}(K,T^*) \cap \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K,T^*)$$

Since $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K,T^*) \subset \mathbf{H}^1_{(\mathcal{F}^*)^\mathfrak{u}}(K,T^*)$, we get that

$$\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{u}}(K,T^*) = \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K,T^*) \qquad \square$$

5.1.3 Chebotarev density theorem

This section is devoted to proving analogues of Propositions 2.2.21 and 2.2.26 in the patched cohomology formalism. In the most general setting, we only need to work under Assumption (R-lim) on the coefficient ring R.

Proposition 5.1.39. Let $\mathfrak{n}$ be a square-free product of ultraprimes, let

$$c_1,\dots,c_s \in \mathbf{H}^1(K^\mathfrak{n}/K,T) \setminus \mathfrak{m}\mathbf{H}^1(K^\mathfrak{n}/K,T)$$

and let

$$d_1,\dots,d_t \in \mathbf{H}^1(K^\mathfrak{n}/K,T^*) \setminus \{0\}$$

Under Assumptions (BI) and (Proj), if $s+t<p$, there exists an ultraprime $\mathfrak{u} \in \mathcal{P}$ such that $\mathrm{loc}_\mathfrak{u}(c_i)$ generates $\mathbf{H}^1_\mathrm{f}(K_\mathfrak{u},T)$ for all $i=1,\dots,s$ and $\mathrm{loc}_\mathfrak{u}(d_j) \neq 0$ for all $j=1,\dots,t$.

Proof. Write

$$c_i = (c_i)_k \in \varprojlim_{k\in\mathbb{N}} \mathbf{H}^1(K^\mathfrak{n}/K,T_k)$$

Since $c_i \notin \mathfrak{m}\mathbf{H}^1(K^\mathfrak{n}/K,T)$, Assumption (Proj) implies that $(c_i)_1$ does not vanish. Fix a sequence $\big((c_i)_1^{(k)}\big)_{k\in\mathbb{N}}$ representing $(c_i)_1$ in

$$\mathbf{H}^1(K^\mathfrak{n}/K,T_1) = \mathcal{U}_k\big(H^1(K^{n_k}/K,T_1)\big)$$

where $(n_k)_{k\in\mathbb{N}}$ is a sequence representing $\mathfrak{n}$. Similarly, there exists $a\in\mathbb{N}$ such that

$$d_j \in \mathbf{H}^1(K^\mathfrak{n}/K,T_a^*)\ \forall j=1,\dots,t$$

Similarly, fix sequences $\big(d_j^{(k)}\big)$ representing d_j in

$$\mathbf{H}^1(K^\mathfrak{n}/K,T_a^*) = \mathcal{U}_k\big(H^1(K^{\mathfrak{n}_k}/K,T_a^*)\big)$$

By Proposition 2.2.21, for every $k\in\mathbb{N}$, we can find a Kolyvagin prime such that $\ell_k \in \mathcal{P}(T_k)$ such that

$$\mathrm{loc}_{\ell_k}\big((c_i)_1^{(k)}\big) \neq 0 \Leftrightarrow \big((c_i)_1^{(k)}\big) \neq 0\ \forall i=1,\dots,s$$
$$\mathrm{loc}_{\ell_k}\big(d_j^{(k)}\big) \neq 0 \Leftrightarrow d_j^{(k)} \neq 0\ \forall j=1,\dots,t$$

Let $\mathfrak{u}$ be the ultraprime represented by the sequence $(\ell_k)_{k\in\mathbb{N}}$. By construction, $\mathfrak{u} \in \mathcal{P}$. Since $(c_i)_1^{(k)} \neq 0$ for $\mathcal{U}$-many k, because it represents a non-zero element in the ultraproduct, then $\mathrm{loc}_{\ell_k}\big((c_i)_1^{(k)}\big) \neq 0$ for $\mathcal{U}$-many k. Hence $\mathrm{loc}_{\mathfrak{u}}((c_i)_1)$ is a non-zero element in $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T_1)$, so $\mathrm{loc}_{\mathfrak{u}}(c_i)$ generates $\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$ for all i. Similarly, we can prove that $\mathrm{loc}_{\mathfrak{u}}(d_j)$ is non-zero for all j. $\square$

In order to apply this proposition, we want to be able to find elements in Selmer groups that are not $\mathfrak{m}$-divisible in the patched cohomology group. When the Selmer group is non-zero, Nakayama's lemma implies that such elements can be found.

Proposition 5.1.12 allows the restatement of Proposition 5.1.39 in the following form.

Corollary 5.1.40. Let $\mathcal{F}_1, \ldots, \mathcal{F}_s$ be cartesian Selmer structures and let $d_1, \ldots, d_t \in \mathbf{H}^1(K, T^*) \setminus \{0\}$, where $s + t < p$. Under Assumptions (BI) and (Proj), if $\mathbf{H}^1_{\mathcal{F}_i}(K, T) \neq 0$ for all i, there exists an ultraprime $\mathfrak{u} \in \mathcal{P}$ such that the maps

$$\mathrm{loc}_{\mathfrak{u}}(c_i) : \mathbf{H}^1_{\mathcal{F}_i}(K, T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$$

are surjective for all $i = 1, \ldots, s$ and $\mathrm{loc}_{\mathfrak{u}}(d_j) \neq 0$ for all $j = 1, \ldots, t$.

Proof. By Nakayama's lemma, there are classes

$$c_i \in \mathbf{H}^1_{\mathcal{F}_i}(K, T) \setminus \mathfrak{m}\mathbf{H}^1_{\mathcal{F}_i}(K, T)$$

Then Proposition 5.1.12 implies that

$$c_i \notin \mathfrak{m}\mathbf{H}^1(K^{\mathfrak{n}}/K, T)$$

where $\mathfrak{n}$ is the square-free product of all the primes in $\Sigma_{\mathcal{F}_1} \cup \cdots \cup \Sigma_{\mathcal{F}_s}$. By Assumption (Proj), the projections are non-zero elements

$$0 \neq (c_i)_1 \in \mathfrak{m}\mathbf{H}^1(K^{\mathfrak{n}}/K, T_1)$$

Hence this corollary is reduced to Proposition 5.1.39 $\square$

One direct consequence of the above is the following corollary.

Corollary 5.1.41. Let $\mathcal{F}$ be a Selmer structure. Under Assumptions (BI) and (Proj), there is some $\mathfrak{n} \in \mathcal{N}$ such that

$$\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*) = 0$$

Proof. Let $\{r_1, \ldots, r_s\}$ be a basis of $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*)[\mathfrak{m}]$ as a k-vector space. By Proposition 5.1.39, we can find ultraprimes $\mathfrak{u}_1, \ldots, \mathfrak{u}_s \in \mathcal{P}$ such that $\mathrm{loc}_{\mathfrak{u}_i}(r_i) \neq 0$. For $\mathfrak{n} := \mathfrak{u}_1 \ldots \mathfrak{u}_s$, we get $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*) = 0$. $\square$

When $\mathbf{H}^1_{\mathcal{F}}(K, T)$ is non-zero, we can construct a vertex $\mathfrak{n} \in \mathcal{N}$ as above but also satisfying an additional property.

Corollary 5.1.42. Let $\mathcal{F}$ be a Selmer structure. Under Assumptions (BI) and (Proj), if $\mathbf{H}^1_{\mathcal{F}}(K, T) \neq 0$, there is some $\mathfrak{n} \in \mathcal{N}$ such that $\mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{n}}}(K, T^*) = 0$ and that, for every $\mathfrak{u} \mid \mathfrak{n}$, there is an equality

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K, T^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}}}(K, T^*)$$

Proof. As we did in Corollary 5.1.41, choose a basis $\{r_1, \ldots, r_s\}$ of $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*)[\mathfrak{m}]$ as a k-vector space. By Corollary 5.1.40, we can find ultraprimes $\mathfrak{u}_i \in \mathcal{P}$ such that $\mathrm{loc}_{\mathfrak{u}_i}(r_i) \neq 0$ and the map

$$\mathrm{loc}_{\mathfrak{u}_i} : \ \mathbf{H}^1_{\mathcal{F}}(K, T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}_i}, T)$$

is surjective. For $\mathfrak{n} := \mathfrak{u}_1 \ldots \mathfrak{u}_s$, we get $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*) = 0$ and, by Lemma 5.1.38, we have that

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u}_i)}(K, T^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}_i}}(K, T^*) \qquad \square$$

When the core rank is positive, we can achieve an analogue of Corollary 5.1.41 with the transverse Selmer group.

Corollary 5.1.43. Let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F}) = 1$. Under Assumptions (BI) and (Proj), there is some $\mathfrak{n} \in \mathcal{N}$ such that $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, T^*) = 0$.

Proof. Similarly, let $\{r_1, \ldots, r_s\}$ be a basis of $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, T^*)[\mathfrak{m}]$ as a k-vector space. Suppose inductively that we have constructed ultraprimes $\mathfrak{u}_1, \ldots, \mathfrak{u}_i$ such that $\mathrm{loc}_{\mathfrak{u}_j}(r_j) \neq 0$ for all $j \leq i$ and

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u}_1 \ldots \mathfrak{u}_i)}(K, T^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}_1 \cdots \mathfrak{u}_i}}(K, T^*)$$

Note that Proposition 5.1.37 implies that $\chi(\mathcal{F}(\mathfrak{u}_1 \ldots \mathfrak{u}_i)) = \chi(\mathcal{F})$ is positive, so, in particular, $\mathbf{H}^1_{\mathcal{F}(\mathfrak{u}_1 \ldots \mathfrak{u}_i)}(K, T) \neq 0$. By Corollary 5.1.40, we can find an ultraprime $\mathfrak{u}_{i+1} \in \mathcal{P}$ such that $\mathrm{loc}_{\mathfrak{u}_{i+1}}(r_{i+1}) \neq 0$ and the map

$$\mathrm{loc}_{\mathfrak{u}_{i+1}} : \ \mathbf{H}^1_{\mathcal{F}(\mathfrak{u}_1 \ldots \mathfrak{u}_i)}(K, T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}_{i+1}}, T)$$

is surjective. By Lemma 5.1.38, we have that

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u}_1\ldots\mathfrak{u}_{i+1})}(K,T^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}_{i+1}}(\mathfrak{u}_1\ldots\mathfrak{u}_i)}(K,T^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}_1\cdots\mathfrak{u}_{i+1}}}(K,T^*)$$

For $\mathfrak{n} := \mathfrak{u}_1\ldots\mathfrak{u}_s$, we get $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K,T^*) = 0$. □

In addition to obtaining a trivial Selmer group by modifying the local condition at certain primes, we will also use a stronger result about reducing the rank of the Selmer group modifying the local condition at only one prime.

Proposition 5.1.44. Let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F}) \geq 0$. Assume that

$$\operatorname{rank}_R \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee > 0$$

Then there is an ultraprime $\mathfrak{u} \in \mathcal{P}$ such that

$$\operatorname{rank}_R \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K,T^*)^\vee = \operatorname{rank}_R \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee - 1$$

Proof. Let T be the R-torsion subgroup of $\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee$. By Lemma 5.1.13, there is an injection

$$\mu:\ \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee_{\text{tors}}/\mathfrak{m} \hookrightarrow \mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee/\mathfrak{m}$$

Choose a k-basis $\{f_1,\ldots,f_s\}$ of $\mu\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee_{\text{tors}}/\mathfrak{m}\big)$ and consider a set $\{d_1,\ldots,d_t\}$ extending it to a basis of $\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee/\mathfrak{m}$.

The localisation $(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee_{/\text{tors}})_{(0)}$ is a finitely-generated $R_{(0)}$ vector space of dimension r coinciding with the rank. Then any set of generators of $\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee_{/\text{tors}}$ contains at least r elements. Since taking quotients by $\mathfrak{m}$ is a right exact functor, then Nakayama's lemma implies that $t \geq r > 0$.

Let $x \in \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)[\mathfrak{m}]$ be the element of the dual basis associated with d_1, i.e., $d_1(x) \neq 0$ and $d_i(x) = 0$ for $i \geq 2$ and $f_i(x) = 0$ for all i.

Note that, since $\chi(\mathcal{F}) \geq 0$, the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K,T) \neq 0$. By Corollary 5.1.40, we can find an ultraprime $\mathfrak{u}$ such that

$$\operatorname{loc}_{\mathfrak{u}}:\ \mathbf{H}^1_{\mathcal{F}}(K,T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},T)$$

is surjective and $\mathrm{loc}^*_{\mathfrak{u}}(x) \neq 0$. Consider then the following diagram

$$\begin{array}{ccc} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T^*)^\vee & \xrightarrow{(\mathrm{loc}^*_{\mathfrak{u}})^\vee} & \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \\ \downarrow & & \downarrow \\ \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T^*)^\vee/\mathfrak{m} & \xrightarrow{(\mathrm{loc}^*_{\mathfrak{u}})^\vee} & \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee/\mathfrak{m} \end{array}$$

The image of the bottom horizontal map contains an element d' satisfying that $d'(x) \neq 0$. Hence the d_1-coordinate of d' in the basis $\{f_1, \ldots, f_s, d_1, \ldots, d_r\}$ is non-zero. In particular, it implies that any lift of d' in $\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee$ is not a torsion element.

Nevertheless, the image top horizontal arrow contains a lift of d', which will be a non-torsion element. In particular, the image has rank one. Since the rank is an additive function, we can conclude that

$$\mathrm{rank}_R\, \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}}}(K, T^*) = \mathrm{rank}_R\, \mathbf{H}^1_{\mathcal{F}^*}(K, T^*) - 1$$

By Lemma 5.1.38 and the surjectivity assumption on $\mathrm{loc}_{\mathfrak{u}}$,

$$\mathrm{rank}_R\, \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K, T^*) = \mathrm{rank}_R\, \mathbf{H}^1_{\mathcal{F}^*}(K, T^*) - 1 \qquad \square$$

5.2 Ultra Kolyvagin systems

The main advantage of using patched cohomology is that we can define Kolyvagin systems directly for T.

Definition 5.2.1. An *ultra Kolyvagin system* for a Selmer structure $\mathcal{F}$ is a collection of classes

$$\kappa = \left\{\kappa_{\mathfrak{n}} \in H^1_{\mathcal{F}(\mathfrak{n})}(K, T) :\ \mathfrak{n} \in \mathcal{N}\right\}$$

satisfying the following relation for every $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ not dividing $\mathfrak{n}$ that

$$\mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}}) = \phi^{\mathrm{fs}}_{\mathfrak{u}} \circ \mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}})$$

for every $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ not dividing $\mathfrak{n}$.

Notation 5.2.2. The set of ultra Kolyvagin systems, which has a natural R-module structure, is denoted by $\mathbf{KS}(T, \mathcal{F})$.

Remark 5.2.3. The same definition can be used to construct the ultra Kolyvagin systems $\mathbf{KS}(T_k, \mathcal{F})$ for the finite quotients of T. It is possible to use the results in [MR04] to $\mathbf{KS}(T_k, \mathcal{F})$, keeping the proofs identical but considering ultraprimes instead of primes. In particular, since R_k is a self-injective ring, then $\mathbf{KS}(T_k, \mathcal{F})$ is a free R_k-module of rank one.

Kolyvagin systems as in Definition 5.2.1 can be used to compute Fitting ideals of Selmer groups when the core rank is one. However, when $\chi(\mathcal{F}) > 1$, the module of Kolyvagin systems is no longer of rank one, so there are no primitive systems. Following the work in [Sak18], [BS21] and [BSS25], we redefine Kolyvagin systems a collection of classes in exterior biduals (see B.4.5) of Selmer groups

$$\kappa_n \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$$

This formalism applies for self-injective, artinian local rings, not necessarily principal. In order to define the Kolyvagin system relations, we will use the maps constructed in Propositions B.4.6, B.4.7 and B.4.8 between bidual maps. These maps were constructed in [BSS25] when the coefficient ring is self-injective and are generalised in this work to exterior biduals of modules over a discrete valuation ring (see Proposition B.4.7) or the Iwasawa algebra (see Proposition B.4.8).

We will now sketch the construction of higher rank Kolyvagin systems. The reader might find interesting §6.3, where this construction is shown in more detail when R is the classical Iwasawa algebra.

For the construction below, we will assume that R is either a discrete valuation ring or the Iwasawa algebra and T is a Galois representation over R. However, the construction will also work when T has coefficients in certain quotients of R or, in more generality, in any self-injective ring R.

Assumption (DVR/IW). R is either a discrete valuation ring or the Iwasawa algebra.

Let $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ not dividing $\mathfrak{n}$. By Proposition 4.3.11, there is an exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}_\mathfrak{u}(\mathfrak{n})}(K, T) \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \longrightarrow \mathbf{H}^1_{\mathrm{s}}(K_\mathfrak{u}, T) \cong R \quad (5.6)$$

Then Corollary B.4.9 induces a map

$$\alpha_{\mathfrak{n},\mathfrak{u}} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \to \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_\mathfrak{u}(\mathfrak{n})}(K, T)$$

Similarly, Proposition 4.3.11 induces another exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}_\mathfrak{u}(\mathfrak{n})}(K, T) \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})} \longrightarrow \mathbf{H}^1_{\mathrm{f}}(K_\mathfrak{u}, T) \cong R$$

We further assume that the last isomorphism is chosen as the composition of the finite-singular map with the isomorphism in (5.6) (see §6.3 for more details). Similarly, Corollary B.4.9 induces a map

$$\beta_{\mathfrak{n},\mathfrak{u}} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,T) \to \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,T)$$

Definition 5.2.4. Let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F})$. A *higher rank Kolyvagin system* is a collection of elements

$$\kappa_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)$$

satisfying that $\alpha_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}}) = \beta_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}})$.

Remark 5.2.5. In comparison with other results in the literature, we fix the rank of the Kolyvagin systems to be the core rank of the Selmer structure $\mathcal{F}$.

Remark 5.2.6. When the core rank is one and R is a discrete valuation ring, Definitions 5.2.1 and 5.2.4 are equivalent since

$$\bigcap^{1} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T) = \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)^{++} = \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)$$

where the last equality follows from R being a discrete valuation ring and the Selmer group being finitely generated. Note that, the compatibility in the choices of the isomorphisms when constructing $\alpha_{\mathfrak{n},\mathfrak{u}}$ and $\beta_{\mathfrak{n},\mathfrak{u}}$ guarantees the equivalence in the Kolyvagin conditions from each definition. When R is the Iwasawa algebra or any self-injective ring, there is only a homomorphism from the module of Kolyvagin systems in Definition 5.2.1 to the one in Definition 5.2.4, not necessarily an isomorphism. However, in the case where R is the Iwasawa algebra, if we further assume that the propagation of $\mathcal{F}$ to $T/(X)$ is also cartesian, then Proposition 6.1.3 is implies that all the Selmer groups are free modules, which result in the equivalence of Definitions 5.2.1 and 5.2.4.

Remark 5.2.7. Definition 5.2.4 can be also used to define ultra Kolyvagin systems for $\mathbf{H}^1_{\mathcal{F}}(K,T_k)$, which is a Selmer group with self-injective coefficient ring R_k (see the notation in Assumption (R-lim)). The module of ultra Kolyvagin systems $\mathbf{KS}(T_k,\mathcal{F})$ satisfies all the results in [BSS25], provided that the propagated Selmer structure $\mathcal{F}$ to T_k satisfies the cartesian hypothesis in loc. cit., about the classical module of Kolyvagin systems $\mathrm{KS}(T,\mathcal{F})$, since all the proofs can be translated to the patched cohomology setting by considering Kolyvagin ultraprimes instead of primes. In particular, the module of ultra Kolyvagin systems $\mathbf{KS}(T_k, \mathcal{F})$ is a free R_k-module of rank one.

In fact, there is a canonical isomorphism between ultra Kolyvagin systems over finite rings and classical Kolyvagin systems. Such isomorphism will be used to study the ultra Kolyvagin systems in $\mathbf{KS}(T, \mathcal{F})$ as a limit of classical Kolyvagin systems.

Theorem 5.2.8. Assume that the propagation of the Selmer structure $\mathcal{F}$ to T_k is cartesian in the sense of Definition 2.2.11. Then there is a canonical isomorphism

$$\mathbf{KS}(T_k, \mathcal{F}) \to \mathrm{KS}(T_k, \mathcal{F}) : \ \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{U}(\mathcal{N})} \mapsto \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{N}}$$

where, only in this theorem, $\mathcal{N}$ denotes the square free product of classical Kolyvagin primes.

The inverse map is is defined by the ultra Kolyvagin system constructed as follows: for every $\mathfrak{n} \in \mathcal{U}(\mathcal{N})$ represented by a sequence $(n_i)_{i \in \mathbb{N}}$, the class $\kappa_{\mathfrak{n}}$ is constructed by patching the classes κ_{n_i}, as explained below.

Proof. Note that, since R_k is finite, the set of square-free products of Kolyvagin ultraprimes is the ultraproduct of square-free products of Kolyvagin primes.

It is clear, that if we have an ultra Kolyvagin system $\kappa = \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{U}(\mathcal{N})}$ in $\mathbf{KS}(T_k, \mathcal{F})$, the subset of $\kappa_{\mathfrak{n}}$ when $\mathfrak{n}$ is a constant ultraprime forms a classical Kolyvagin system in $\mathrm{KS}(T_k, \mathcal{F})$.

We can now outline the construction of the inverse map. Assume that $\kappa = \{\kappa_n\}_{n \in \mathcal{N}} \in \mathrm{KS}(T_k, \mathcal{F})$ is a classical Kolyvagin system and let $\mathfrak{n}$ be a square-free product of Kolyvagin ultraprimes represented by the sequence $(n_i)_{i \in \mathbb{N}}$. By Corollary B.6.2, we have that

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k) = \mathcal{U}_i \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(n_i)}(K, T_k) \right)$$

Then the sequence $(\kappa_{n_i})_{i \in \mathbb{N}}$ represents an element in the exterior bidual $\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k)$, that will be defined as $\kappa_{\mathfrak{n}}$. Note that the construction of $\kappa_{\mathfrak{n}}$ is independent of the choice of the representative sequence.

For every ultraprime $\mathfrak{u} = (\ell_i)_{i \in \mathbb{N}}$ dividing $\mathfrak{n}$, the sequences $(\phi_{\ell_i})_{i \in \mathbb{N}}$ and $(\psi_{\ell_i})_{i \in \mathbb{N}}$ represent elements

$$\phi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathfrak{u}}(K_{\mathfrak{u}}, T_k)^+, \quad \psi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, T_k)^+$$

respectively, satisfying that $\phi_{\mathfrak{u}} = \phi_{\mathfrak{u}}^{\mathrm{fs}} \circ \psi_{\mathfrak{u}}$. We will use those to compute the Kolyvagin system relations.

By Proposition B.6.4, we have that

$$\alpha_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{u}}) = (\alpha_{n_i,\ell_i}(\kappa_{n_i})), \quad \beta_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}}) = (\beta_{n_i,\ell_i}(\kappa_{n_i\ell_i}))$$

where we are using the following identification, given by Corollary B.6.2:

$$\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,T) = \mathcal{U}_i \left(\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\ell_i}(n_i)}(K,T) \right)$$

Since κ_{n_i} and $\kappa_{n_i\ell_i}$ satisfy the Kolyvagin system relation $\alpha_{n_i,\ell_i}(\kappa_{n_i}) = \beta_{n_i,\ell_i}(\kappa_{n_i\ell_i})$ for all $i \in \mathbb{N}$, then

$$\alpha_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{u}}) = (\beta_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}}))$$

Then this construction gives a well defined map

$$\mathrm{KS}(T_k, \mathcal{F}) \to \mathbf{KS}(T_k, \mathcal{F})$$

It is a direct computation to check that the composition

$$\mathrm{KS}(T_k, \mathcal{F}) \to \mathbf{KS}(T_k, \mathcal{F}) \to \mathrm{KS}(T_k, \mathcal{F})$$

is the identity map. Indeed, when $\mathfrak{n}$ is equivalent to a constant sequence (n), then, by construction, of $\kappa_{\mathfrak{n}}$ is the patching of a sequence $(\kappa_{n_i})_{i\in\mathbb{N}}$, where $n_i = n$ for $\mathcal{U}$-many i. Then $\kappa_{\mathfrak{n}} = \kappa_n$.

Since both $\mathrm{KS}(T_k, \mathcal{F})$ and $\mathbf{KS}(T_k, \mathcal{F})$ are free R_k-modules of rank one, the above maps are in fact isomorphisms. $\square$

5.2.1 Projection maps of biduals

We have seen how we can identify ultra Kolyvagin systems with finite coefficients with classical ones. In this sections, we want to generalise these ideas in order to describe the ultra Kolyvagin systems $\mathbf{KS}(T, \mathcal{F})$ as a limit of Kolyvagin systems with finite coefficients. A central step of this process is the description of the biduals of a Selmer group as the limit

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \cong \varprojlim_{k\in\mathbb{N}} \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k) \right)$$

This description is only true when the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K, T)$ is a free module, what will lead to Assumption (Free) below. However, such assumption is not very restrictive, since it should hold under all the other assupmptions previously made.

First, we need to say what are the transition maps in the limit.

Lemma 5.2.9. Let $k \in \mathbb{N}$. Then there is a canonical map

$$\bigcap_{R_{k+1}}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_{k+1}) \to \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$

Proof. By Remark 5.1.1, Corollary B.5.3 provides a map

$$\bigcap_{R_{k+1}}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_{k+1}) \to \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_{k+1}) \otimes_{R_{k+1}} R_k$$

The functoriality of the Selmer grous induces a map

$$\mathbf{H}^1_{\mathcal{F}}(K, T_{k+1}) \otimes_{R_{k+1}} R_k \to \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$

And the proof is completed by the functoriality of the exterior bidual shown in Propositionn B.4.10. $\square$

Hence the above inverse limit is well defined. Now we want to construct a map from the exterior bidual of the profinite Selmer group.

Lemma 5.2.10. Let $\mathcal{F}$ be a cartesian Selmer structure on T such that $\mathbf{H}^1_{\mathcal{F}}(K, T)$ is a free R-module. For every $k \in \mathbb{N}$, there is a canonical map

$$\bigcap_{R}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T) \to \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$

Proof. Since $\mathbf{H}^1_{\mathcal{F}}(K, T)$ is a free R-module, there is an identification

$$\left(\bigcap_{R}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T) \right) \otimes R_k = \bigcap_{R_k}^{\chi(\mathcal{F})} \big(\mathbf{H}^1_{\mathcal{F}}(K, T) \otimes R_k \big)$$

The lemma follows from the composition with the following map, given by Proposition B.4.10:

$$\bigcap_{R_k}^{\chi(\mathcal{F})} \big(\mathbf{H}^1_{\mathcal{F}}(K, T) \otimes R_k \big) \to \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, T_k)$$ $\square$

Corollary 5.2.11. Let $\mathcal{F}$ be a cartesian Selmer structure on T such that $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free R-module. Then there is a canonical map

$$\bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \to \varprojlim_{k\in\mathbb{N}} \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k)$$

Lemma 5.2.12. Let $\mathcal{F}$ be a cartesian Selmer structure on T such that $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free R-module. Then the map in Corollary 5.2.11 is an isomorphism

$$\bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \cong \varprojlim_{k\in\mathbb{N}} \left(\bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k) \right)$$

Proof. Since $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free, finitely generated R-module, there is an isomorphism

$$\bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \cong \varprojlim_{k\in\mathbb{N}} \left(\bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \otimes_R R_k \right)$$

By Proposition 5.1.15 and the left-exactness of the exterior bidual and the inverse limit, the following map is injective:

$$\varprojlim_{k\in\mathbb{N}} \left(\bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \otimes_R R_k \right) \hookrightarrow \varprojlim_{k\in\mathbb{N}} \left(\bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k) \right)$$

For the surjectivity, let

$$(c_k)_{k\in\mathbb{N}} \in \varprojlim_{k\in\mathbb{N}} \left(\bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k) \right)$$

Then, for every $k \in \mathbb{N}$ and every $j \geq k$,

$$c_k \in \operatorname{Im}\left(\bigcap_{R_j}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_j) \to \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k) \right)$$

Then

$$c_k \in \bigcap_{R_k}^{\chi(\mathcal{F})} \Big(\operatorname{Im}\big(\mathbf{H}^1_{\mathcal{F}}(K,T_j) \to \mathbf{H}^1_{\mathcal{F}}(K,T_k) \big) \Big)$$

Since $\mathbf{H}^1_{\mathcal{F}}(K,T) = \varprojlim_{k\in\mathbb{N}} \mathbf{H}^1_{\mathcal{F}}(K,T_k)$ by Proposition 4.3.6, then

$$c_k \in \bigcap_{R_k}^{\chi(\mathcal{F})} \mathrm{Im}\big(\mathbf{H}^1_{\mathcal{F}}(K,T) \to \mathbf{H}^1_{\mathcal{F}}(K,T_k)\big)$$

which completes the proof of the surjectivity. □

5.2.2 Projection maps of Kolyvagin systems

In this section, we want to express the ultra Kolyvagin systems as the limit of classical Kolyvagin systems with finite coefficients. In order to do that, the results in [BSS25] are required, so we need to impose that the propagated Selmer structure on T_k satisfies the cartesian condition from Definition 2.2.11.

Assumption (FC). We assume that, for all $k \in \mathbb{N}$, the propagation of $\mathcal{F}$ to T_k is cartesian, meaning that, for every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, the following map, induced by the multiplication by a generator of $R_k[\mathfrak{m}]$, is injective:

$$\mathbf{H}^1_{\mathcal{F}}(K,T_1) \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T_k)$$

In addition, we assume that the core ranks of the propagated Selmer structures coincide with $\chi(\mathcal{F})$.

Remark 5.2.13. When R is a discrete valuation ring, Assumption (FC) holds whenever $\mathcal{F}$ is cartesian (see Propositions 5.3.5 and 5.3.7). In case R is the Iwasawa algebra, Assumption (FC) if both $\mathcal{F}$ and its propagation to $T/(X)$ are cartesian (see Propositions 6.1.16 and 6.1.17).

Proposition 5.2.14. ([BSS25, §5.5]) For every $n \in \mathbb{N}$, there is a canonical surjective map

$$\mathbf{KS}(T_k,\mathcal{F}) \to \mathbf{KS}(T_{k-1},\mathcal{F})$$

Proof. It is a combination of Theorem 5.2.8 and the argument in [BSS25, §5.5].

We will sketch this construction: if $\mathfrak{n} \in \mathcal{N}(T_k)$, we can consider the image of $\kappa_{\mathfrak{n}}$ under the projection map

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T_k) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T_{k-1}) \tag{5.7}$$

constructed in [BSS25, Corollary 2.7]. This process can be used to construct Kolyvagin classes in a subset of $\mathcal{N}(T_{k-1})$ satisfying the relations characteristic of Kolyvagin systems. They satisfy the Kolyvagin system relations, due to Propositions B.4.10 and B.5.4.

It can be shown that this collection can be uniquely extended to a Kolyvagin system in $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_{k-1})$. The argument behind this extensibility property comes from the fact that $\mathcal{N}(\mathcal{P}(T_{k-1}))$ contains a core vertex (see Definition 6.1.8 below), which controls the module of Kolyvagin systems (see [BSS25, Theorem 5.20] or Theorem 6.3.5 below). $\square$

Remark 5.2.15. The projection maps in Proposition 5.2.14 permits to consider the inverse limit of Kolyvagin systems

$$\overline{\mathbf{KS}}(T, \mathcal{F}) = \varprojlim_{k \in \mathbb{N}} \mathrm{KS}(T_k, \mathcal{F})$$

Lemma 5.2.12 allows us to compare the module of ultra Kolyvagin systems $\mathbf{KS}(T, \mathcal{F})$ with the limit constructed in Remark 5.2.15. However, it requires the Selmer groups $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$ to be free R-modules for all $\mathfrak{n} \in \mathcal{N}$.

Assumption (Free). Assume that, for every Selmer structure $\mathcal{F}$, the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K, T)$ is a free R-module.

Remark 5.2.16. When R is a discrete valuation ring, the freeness of $\mathbf{H}^1_{\mathcal{F}}(K, T)$ follows from Proposition 5.1.4. On the other hand, when R is the Iwasawa algebra, Assumption (Free) will be proven in Proposition 6.1.15 to be implied by the cartesian condition in both $\mathcal{F}$ and its propagation to $T/(X)$.

Lemma 5.2.17. Suppose Assumption (Free) holds. For every $k \in \mathbb{N}$, there is a canonical projection map

$$\mathbf{KS}(T, \mathcal{F}) \to \mathbf{KS}(T_k, \mathcal{F})$$

Proof. Let $n \in \mathcal{N}$ and $k \in \mathbb{N}$. Since $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$ is a free R-module, we can define $\kappa_{\mathfrak{n}}^{(k)}$ as the image of $\kappa_{\mathfrak{n}}$ under the map constructed in Lemma 5.2.10.

We want to see that the collection $\{\kappa_{\mathfrak{n}}^{(k)}\}_{\mathfrak{n} \in \mathcal{N}(T)}$ satisfies the Kolyvagin system relations. We will denote this fact by stating that

$$\{\kappa_{\mathfrak{n}}^{(k)}\}_{\mathfrak{n} \in \mathcal{N}(T)} \in \mathbf{KS}(T_k, \mathcal{F}, \mathcal{N}(T))$$

Proposition B.4.10 guarantees that the newly constructed classes satisfy the Kolyvagin system relations.

Similarly, there is a canonical map

$$\mathbf{KS}(T_k, \mathcal{F}) \to \mathbf{KS}(T_k, \mathcal{F}, \mathcal{N}(T))$$

obtained by considering the classes $\kappa_{\mathfrak{n}}$ only when $\mathfrak{n} \in \mathcal{N}(T)$. We can see that this map is an isomorphism, since all the arguments using the Chebotarev density theorem in [BSS25] can be used to construct a Kolyvagin ultraprime in $\mathcal{P}(T)$ instead of a Kolyvagin prime in $\mathcal{P}(T_k)$. In particular, we can show that there exists a core vertex in $\mathcal{N}(T)$, which implies the injectivity. In addition, the arguments in [BSS25, Theorem 5.20] show that $\mathrm{KS}(T_k, \mathcal{F}, \mathcal{N}(T))$ is a free R_k-module of rank one, which leads to the surjectivity.

By considering the composition with the inverse map, there is a projection map

$$\mathbf{KS}(T, \mathcal{F}) \to \mathbf{KS}(T_k, \mathcal{F}) \qquad \square$$

The projection maps are compatible in the sense that they induce a map to the inverse limit in Proposition 5.2.15.

Corollary 5.2.18. Suppose Assumption (Free) holds. Then there is a canonical map

$$\mathbf{KS}(T, \mathcal{F}) \to \overline{\mathbf{KS}}(T, \mathcal{F})$$

We can show that the above map is indeed an isomorphism.

Theorem 5.2.19. Under Assumptions (DVR/IW) (Free), the map in Corollary 5.2.18

$$\mathbf{KS}(T, \mathcal{F}) \to \overline{\mathbf{KS}}(T, \mathcal{F}) \tag{5.8}$$

is an isomorphism.

The above theorem permits the computation of the module of ultra Kolyvagin systems.

Corollary 5.2.20. The ultra Kolyvagin systems $\mathbf{KS}(T, \mathcal{F})$ form a free R-module of rank one.

Remark 5.2.21. There exists a different proof of Corollary 5.2.20 obtained after developing a theory of ultra Kolyvagin systems. It will be done below (see Corollary 6.3.15) when the coefficient ring R is the Iwasawa algebra, but the same proof applies to other coefficient rings like discrete valuation rings.

Proof of Theorem 5.2.19. Assume that $\kappa \in \mathbf{KS}(T, \mathcal{F})$ is in the kernel of this map. For every $\mathfrak{n} \in \mathcal{N}(T)$, we have by Theorem 5.2.19 that

$$\kappa_{\mathfrak{n}} \in \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) = \varprojlim_{k \in \mathbb{N}} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k)$$

Since the projection of $\kappa_{\mathfrak{n}}$ to every Selmer group with finite coefficients vanishes, the relation shows $\kappa_{\mathfrak{n}} = 0$. Hence, the map in (5.8) is injective.

For the surjectivity, let

$$\overline{\kappa} = (\kappa^{(k)})_{k \in \mathbb{N}} \in \overline{\mathbf{KS}}(T, \mathcal{F})$$

Note that $\mathcal{N}(T) \subset \mathcal{N}(T_k)$ for all $k \in \mathbb{N}$. Then, for every $\mathfrak{n} \in \mathcal{N}(T_k)$, the collection $\kappa_{\mathfrak{n}}^{(k)}$ is compatible under projection maps, so Lemma 5.2.12 can be used to construct an element

$$\kappa_{\mathfrak{n}} = (\kappa_{\mathfrak{n}}^{(k)})_{k \in \mathbb{N}} \in \varprojlim_{k \in \mathbb{N}} \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k) \right) = \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$$

Then the collection $\kappa := \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathcal{N}(T)}$ satisfies the Kolyvagin system relations, since they follow by taking limits from the Kolyvagin relations for the classes in the finite cohomology groups and the maps from Propositions B.4.7 and B.4.8 are compatible with the inverse limits.

It is clear than the map in (5.8) sends κ to $(\kappa^{(k)})_{k \in \mathbb{N}}$, which concludes the proof of the surjectivity. $\square$

5.2.3 Projection maps of theta ideals

Analogously to the classical case, ultra Kolyvagin systems interact with Selmer groups via theta ideals. Below it is a restatement of Definition 3.1.8.

Definition 5.2.22. Let $\kappa \in \mathbf{KS}(\mathcal{F})$ be an ultra Kolyvagin system. The theta ideals of κ are defined as

$$\mathbf{\Theta}_i(\kappa) := \sum_{\mathfrak{n} \in \mathcal{N}_i(T)} \operatorname{ind} \left(\kappa_{\mathfrak{n}}, \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \right)$$

Definition 5.2.22 also applies to the ultra Kolyvagin systems with finite coefficients in $\mathbf{KS}(T_k, \mathcal{F})$. In that case, they can be directly compared with the ideals from Definition 3.1.8.

Lemma 5.2.23. Let $\kappa^{(k)} \in \mathbf{KS}(T_k, \mathcal{F})$ and let $\mathfrak{n} = (n_i)_{i\in\mathbb{N}} \in \mathcal{N}(T_k)$. For $\mathcal{U}$-many i, the following equality holds

$$\operatorname{ind}\left(\kappa_{\mathfrak{n}}^{(k)}, \bigcap_{R_k}^{\chi(\mathcal{F})} H^1_{\mathcal{F}(\mathfrak{n})}(K, T_k)\right) = \operatorname{ind}\left(\kappa_{n_i}^{(k)}, \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(n_i)}(K, T_k)\right)$$

Proof. By Definition 4.2.1 and B.6.2, we have that

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k) = \mathcal{U}\left(\bigcap^{\chi(\mathcal{F})} H^1_{\mathcal{F}(n_i)}(K, T_k)\right)$$

By Theorem 5.2.8, $\kappa_{\mathfrak{n}}$ can be represented by the sequence (κ_{n_i}) in this ultraproduct. By Corollary 4.1.3, for $\mathcal{U}$-many i, there is an isomorphism

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T_k) \cong \bigcap^{\chi(\mathcal{F})} H^1_{\mathcal{F}(n_i)}(K, T_k)$$

identifying $\kappa_{\mathfrak{n}}$ with κ_{n_i}, which completes the proof of the lemma. □

Corollary 5.2.24. Let $\kappa^{(k)} \in \mathrm{KS}(T_k, \mathcal{F}) = \mathbf{KS}(T_k, \mathcal{F})$. Then the classical and ultra theta ideals, constructed in Definitions 3.1.8 and 5.2.22 coincide.

Proof. It is clear that

$$\Theta_i(\kappa^{(k)}) \subset \boldsymbol{\Theta}_i(\kappa^{(k)})$$

since the constant Kolyvagin ultraprimes are a subset of the Kolyvagin ultraprimes. Conversely, we need to show that

$$\operatorname{ind}\Big(\kappa_{\mathfrak{n}}, \mathbf{H}^1_{\mathcal{F}(n)}(K, T)\Big) \subset \Theta_i(\kappa^{(k)})$$

However, it follows from Lemma 5.2.23, since there exists a square free product of i classical Kolyvagin primes n such that

$$\operatorname{ind}(\kappa_{\mathfrak{n}}) = \operatorname{ind}(\kappa_n) \qquad \square$$

We now want to give a description of the indices of a Kolyvagin class $\kappa_{\mathfrak{n}}$ in the profinite Selmer group in terms of the indices of the classes in the finite Selmer groups.

Lemma 5.2.25. Let $\mathcal{F}$ be a Selmer structure such that the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free R-module. Let $\alpha \in \bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)$ and let $\overline{\alpha} \in \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k)$ be its image under the projection map in Lemma 5.2.10. Then

$$\operatorname{ind}\left(\alpha, \bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)\right) + I_k = \operatorname{ind}\left(\overline{\alpha}, \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k)\right)$$

where I_k is the kernel of the map $R \twoheadrightarrow R_k$ and we are identifying the ideals of R_k with the ideals of R containing I_k.

Proof. Since $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free module, we can consider the class

$$\alpha + I_k \in \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)\right) \otimes_R R_k = \bigcap^{\chi(\mathcal{F})} \big(\mathbf{H}^1_{\mathcal{F}}(K,T) \otimes_R R_k\big)$$

By construction, it is clear that

$$\operatorname{ind}(\alpha) + I_k = \operatorname{ind}(\alpha + I_k)$$

Then $\overline{\alpha}$ is the image of $\alpha + I^k$ under the injective map (see Proposition 5.1.15). Since the exterior bidual of self-injective rings is a left exact functor, we can conclude that

$$\bigcap^{\chi(\mathcal{F})} \big(\mathbf{H}^1_{\mathcal{F}}(K,T) \otimes_R R_k\big) \hookrightarrow \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k)$$

By Remark 3.1.6, we can conclude that

$$\operatorname{ind}(\overline{\alpha}) = \operatorname{ind}(\alpha + I_k) = \operatorname{ind}(\alpha) + I_k \qquad \square$$

We will apply the previous result to Kolyvagin classes.

Corollary 5.2.26. Suppose Assumption (Free) holds. Consider an ultra Kolyvagin system $\kappa \in \mathbf{KS}(T,\mathcal{F})$ and let $\kappa^{(k)} \in \mathbf{KS}(T_k,\mathcal{F})$ be its projection (see Lemma 5.2.17). Then, for every $\mathfrak{n} \in \mathcal{N}$

$$\operatorname{ind}\left(\kappa_{\mathfrak{n}}, \bigcap_R^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)\right) + I_k = \operatorname{ind}\left(\kappa^{(k)}_{\mathfrak{n}}, \bigcap_{R_k}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T_k)\right)$$

This result proves a subset relation between the theta ideals. Since $\mathcal{N}_i(T) \subset \mathcal{N}_i(T_k)$, the following corollary holds:

Corollary 5.2.27. Let $\kappa \in \mathbf{KS}(T, \mathcal{F})$ be an ultra Kolyvagin system and let $\kappa^{(k)} \in \mathbf{KS}(T_k, \mathcal{F})$ be its projection (see Lemma 5.2.17). Then

$$\mathbf{\Theta}_i(\kappa) \subset \mathbf{\Theta}_i(\kappa^{(k)})$$

The notion of theta ideal can be extended to limits of Kolyvagin systems in $\mathbf{KS}(T_k, \mathcal{F})$.

Proposition 5.2.28. Let $\overline{\kappa} = (\kappa^{(k)})_{k\in\mathbb{N}} \in \overline{\mathbf{KS}}(T)$. For every $k \in N$ and $i \in \mathbb{Z}_{\geq 0}$, we have that

$$\mathbf{\Theta}_i\big(\kappa^{(k+1)}\big) \otimes_{R_{k+1}} R_k \subset \mathbf{\Theta}_i\big(\kappa^{(k)}\big)$$

Therefore, the inverse limit

$$\mathbf{\Theta}_i(\overline{\kappa}) := \varprojlim_{k\in\mathbb{N}} \mathbf{\Theta}_i\big(\kappa^{(k)}\big)$$

is well-defined.

Proof. Let $\mathfrak{n} \in \mathcal{N}(T_{k+1})$ be a constant ultraprime. By [BSS25, Theorem 5.2] (generalising Lemma 3.2.4), we have that

$$\mathrm{ind}\big(\kappa^{(j)}_{\mathfrak{n}}, \mathbf{H}^1_{\mathcal{F}}(K, T_j)\big) = \mathrm{ind}(\kappa^{(j)}, \mathbf{KS}(T_j, \mathcal{F}))\mathrm{Fitt}^0_{R_j}\big(\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K, T_j^*)^\vee\big)$$

for all $j \leq k+1$. Note that the compatibility relation between the Kolyvagin systems implies that

$$\mathrm{ind}(\kappa^{(k)}, \mathbf{KS}(T_k, \mathcal{F})) = \mathrm{ind}(\kappa^{(k+1)}, \mathbf{KS}(T_{k+1}, \mathcal{F})) + I_k$$

In addition, Proposition 5.1.17 implies that

$$\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K, T_{k+1}^*)^\vee \otimes_{R_{k+1}} R_k = \mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K, T_k^*)^\vee$$

Then

$$\mathrm{Fitt}^0_{R_j}\big(\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K, T_k^*)^\vee\big) = \mathrm{Fitt}^0_{R_j}\big(\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K, T_{k+1}^*)^\vee\big) + I_k$$

Taking all these identities into account, we obtain that

$$\mathrm{ind}\big(\kappa^{(k)}_{\mathfrak{n}}, \mathbf{H}^1_{\mathcal{F}}(K, T_k)\big) = \mathrm{ind}\big(\kappa^{(k+1)}_{\mathfrak{n}}, \mathbf{H}^1_{\mathcal{F}}(K, T_{k+1})\big) + I_k$$

Using Corollary 5.2.24 and the inclusion $\mathcal{N}(T_{k+1}) \subset \mathcal{N}(T_k)$, we can show that

$$\mathbf{\Theta}_i\big(\kappa^{(k+1)}\big) \otimes_{R_{k+1}} R_k \subset \mathbf{\Theta}_i\big(\kappa^{(k)}\big)$$ □

This inclusion relation allows us to consider the inverse limit of the theta ideals.

Definition 5.2.29. Let $\overline{\kappa} = (\kappa^{(k)})_{k\in\mathbb{N}} \in \overline{\mathbf{KS}}(T)$. Its theta ideal is defined as

$$\mathbf{\Theta}_i(\overline{\kappa}) := \varprojlim_{k\in\mathbb{N}} \mathbf{\Theta}_i\big(\kappa^{(k)}\big)$$

Theorem 5.2.30. Let $\kappa \in \mathbf{KS}(T, \mathcal{F})$ and let $\overline{\kappa} = (\kappa^{(k)})_{k\in\mathbb{N}} \in \overline{\mathbf{KS}}(T, \mathcal{F})$ be its image under the isomorphism in Theorem 5.2.19. For all $i \in \mathbb{Z}_{\geq 0}$, the following holds:

$$\mathbf{\Theta}_i(\kappa) = \mathbf{\Theta}_i(\overline{\kappa})$$

Proof. By Corollary 5.2.27 and taking the limit, we can deduce that

$$\mathbf{\Theta}_i(\kappa) \subset \mathbf{\Theta}_i(\overline{\kappa})$$

Conversely, let $a = (a_k) \in \varprojlim_{k\in\mathbb{N}} \mathbf{\Theta}_i(\kappa^{(k)}) \subset R$. By Corollary 5.2.24, and the construction of theta ideals, there are square-free products of i Kolyvagin primes $n_{k,1}, \ldots, n_{k,s_k} \in \mathcal{N}_i(T_k)$ such that

$$a_k \in \sum_{i=1}^{s_k} \mathrm{ind}(\kappa^{(k)}_{n_{k,i}})$$

In an attempt to simplify notation, we denote $n_{k,i} := n_{k,s_k}$ for every $i \geq s_k$.

For every $i \in \mathbb{N}$ $\mathfrak{n}_i$ be the square-free product of ultraprimes represented by the sequence $(n_{k,i})_{k\in\mathbb{N}}$. By Lemma 5.2.23 and Corollary 5.2.26, for every fixed i and $\mathcal{U}$-many k, we have that

$$\mathrm{ind}(\kappa_{\mathfrak{n}_i}) + I_k = \mathrm{ind}(\kappa^{(k)}_{n_{j,i}})$$

By the compatibility relation of the different Kolyvagin systems in the tower, we have that, for every $j \geq k$,

$$a_k \in \sum_{i=1}^{s_k} \mathrm{ind}(\kappa^{(j)}_{n_{j,i}}) + I_k$$

where we are again using the equivalence of ideals of R_k and ideals of R containing I_k. That implies that

$$a_k \in \sum_{i=1}^{s_k} \mathrm{ind}(\kappa_{\mathfrak{n}_i}) + I_k \subset \mathbf{\Theta}_i(\kappa) + I_k$$

Since R is compact in the profinite topology, every (finitely generated) ideal is also compact, so closed. Since I_k is a basis of neighbourhoods of the identity and

$$(a + I_k) \cap \mathbf{\Theta}_i(\kappa) \neq \emptyset \ \forall i \in \mathbb{N}$$

Then the closure of $\mathbf{\Theta}_i(\kappa)$ implies that it contains a. □

We can now use this limit to compare the theta ideals of ultra Kolyvagin systems with the Fitting ideals of the Selmer group.

Corollary 5.2.31. Let $\mathcal{F}$ be a cartesian Selmer structure of positive core rank satisfying Assumptions (FC) and (Free) and let $\kappa \in \mathbf{KS}(T, \mathcal{F})$ be a primitive ultra Kolyvagin system. Then

$$\Theta_i(\kappa) \subset \mathrm{Fitt}^i_R\big(\mathbf{H}^1(K, T^*)^\vee\big)$$

Proof. Let $\overline{\kappa} \in \overline{\mathrm{KS}}(T, \mathcal{F})$ the image of κ under the isomorphism in Theorem 5.2.19. By [BSS25, Theorem 5.2],

$$\Theta_i(\overline{\kappa}) \subset \varprojlim_{k \in N} \mathrm{Fitt}^i_{R_k}\big(\mathbf{H}^1(K, T^*_k)^\vee\big)$$

By Proposition 5.1.17, we can conclude that

$$\Theta_i(\kappa) \subset \mathrm{Fitt}^i_R\big(\mathbf{H}^1(K, T^*)^\vee\big)$$ □

5.3 Discrete valuation rings

In this section, we particularise the case where the coefficient ring is a discrete valuation ring. That is arguably the simplest class of rings satisfying Assumption (R-lim). Particularly, we will show that every cartesian Selmer structure satisfies Assumption (FC).

There are no new results in this case, since these Selmer groups have already been computed using the classical theory of Kolyvagin systems, using the limit description in Theorem 5.2.19.

Assumption (DVR). Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p.

In this case, the profinite Selmer groups are free, while the dual discrete Selmer group has a richer structure that can be studied by Kolyvagin systems.

Remark 5.3.1. Note that Proposition 5.1.4 and the structure theorem of finitely generated modules over principal rings imply that $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free R-module for all Selmer structures $\mathcal{F}$.

When R is a discrete valuation ring, Assumption (Proj) is a consequence of Proposition 4.2.6.

Proposition 5.3.2. When R is a discrete valuation ring, Assumption (Proj) holds.

Proof. It is an immediate consequence of the long exact sequence in Proposition 4.2.6 applied to the short exact sequence

$$0 \longrightarrow T \xrightarrow{\pi} T \longrightarrow T_1 \longrightarrow 0 \qquad \square$$

where π represents a generator of the maximal ideal of R. The combination of Corollary 5.1.14 and Proposition 5.1.15 and 5.1.14 imply that the following map is injective for all $k \in \mathbb{Z}_{\geq 0}$:

$$\mathbf{H}^1_{\mathcal{F}}(K,T)/\mathfrak{m}^k \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T/\mathfrak{m}^k)$$

However, when R is a discrete valuation ring, there is a straightforward argument and the cokernel of these maps can be also controlled. The proof of the following proposition is identical to the one of [MR04, Lemma 3.7.1].

Proposition 5.3.3. ([MR04, Lemma 3.7.1]) Let $\mathcal{F}$ be a cartesian Selmer structure. For every $k \in \mathbb{N}$, the following maps are injective and their cokernels C_k are bounded uniformly on k:

$$f : \mathbf{H}^1_{\mathcal{F}}(K,T)/\mathfrak{m}^k \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T/\mathfrak{m}^k)$$

Proof. Let $\mathfrak{n}$ be the square-free product of the ultraprimes in $\Sigma_{\mathcal{F}}$ and let π be a generator of $\mathfrak{m}$. Consider the exact sequence

$$0 \longrightarrow T \xrightarrow{\pi^k} T \longrightarrow T/\mathfrak{m}^k \longrightarrow 0$$

Then the long exact sequence from Proposition 4.2.6 induces an injection

$$\mathbf{H}^1(K^{\mathfrak{n}}/K,T)/\mathfrak{m}^k \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K,T/\mathfrak{m}^k)$$

Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbf{H}^1_{\mathcal{F}}(K,T)/\mathfrak{m}^k \arrow[r, "f"] \arrow[d, hook, "g_1"] & \mathbf{H}^1_{\mathcal{F}}(K,T/\mathfrak{m}^k) \arrow[r, two heads] \arrow[d, hook, "g_2"] & C_k \arrow[d, "g_3"] \\
\mathbf{H}^1(K^{\mathfrak{n}}/K,T)/\mathfrak{m}^k \arrow[r, tail] & \mathbf{H}^1(K^{\mathfrak{n}}/K,T/\mathfrak{m}^k) \arrow[r, two heads] & \mathbf{H}^2(K^{\mathfrak{n}}/K,T)[\mathfrak{m}]
\end{tikzcd}$$

From this diagram, it follows that f is an injective map. In addition, the snake lemma produces an isomorphism

$$\ker(g_3) = \ker\big(\operatorname{coker}(g_1) \to \operatorname{coker}(g_2)\big)$$

In order to compute the last kernel, consider the exact sequence

$$0 \longrightarrow \frac{\mathbf{H}^1(K^{\mathfrak{n}}/K, T)}{\mathbf{H}^1_{\mathcal{F}}(K, T)} \longrightarrow \prod_{\mathfrak{u} \in \Sigma_{\mathcal{F}}} \frac{\mathbf{H}^1(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)} \longrightarrow D \longrightarrow 0$$

where D is a finitely generated module, defined as the cokernel of the first map. Since, for every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, the map

$$\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T)/\mathfrak{m}^k \hookrightarrow \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)$$

is injective, the cartesian condition of $\mathcal{F}$ and another application of the snake lemma implies that

$$\ker(g_3) = \ker\big(\operatorname{coker}(g_1) \to \operatorname{coker}(g_2)\big) = D[\mathfrak{m}^k]$$

Hence, for all k, the cardinality of C_k is bounded above by the finite quantity

$$\#C_k \leq \#\mathbf{H}^2(K^{\mathfrak{n}}, T)_{\mathrm{tors}} + \#D_{\mathrm{tors}}$$

which is independent of k. □

Our next goal is to prove an analogue of Proposition 2.2.16 for patched Selmer groups. We start provig the following generalisation of [Wil95, Proposition 1.6], whose proof appears in [Swe22].

Lemma 5.3.4. ([Swe22, Proposition 2.6.6]) Let $k \in \mathbb{N}$, let $\mathcal{F}$ be a patched Selmer structure and let $\mathfrak{n}$ be a product of ultraprimes divisible by all the primes in $\Sigma_{\mathcal{F}}$. Then the following formula holds

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \frac{\#\mathbf{H}^0(K^{\mathfrak{n}}/K, T/\mathfrak{m}^k)}{\#\mathbf{H}^0(K^{\mathfrak{n}}/K, T^*[\mathfrak{m}^k])} \prod_{\mathfrak{u} | \mathfrak{n}} \frac{\#\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)}{\#\mathbf{H}^0(K_{\mathfrak{u}}, T/\mathfrak{m}^k)}$$

The previous lemma can be used to compare the cardinalities of the Selmer group and its dual.

Proposition 5.3.5. Let $\mathcal{F}$ be a patched cartesian Selmer structure. If Assumption (BI) holds, then

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \chi(\mathcal{F}) \cdot \big(\#(R/\mathfrak{m})\big)^k \quad \forall k \in \mathbb{N}$$

Proof. We start by showing that the quotient

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])}$$

is linear on k. By Remark 5.1.3 and Lemma 5.3.4, we have that

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \prod_{\mathfrak{u}|\mathfrak{n}} \frac{\#\mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^k)}{\#\mathbf{H}^0(K_\mathfrak{u}, T/\mathfrak{m}^k)} \tag{5.9}$$

The linearity of the last sequence follows from the next exact sequence for every ultraprime $\mathfrak{u}$ and every indices $i, j \in \mathbb{N}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^0(K_\mathfrak{u}, T/\mathfrak{m}^i) \arrow[r] & \mathbf{H}^0(K_\mathfrak{u}, T/\mathfrak{m}^{i+j}) \arrow[r] & \mathbf{H}^0(K_\mathfrak{u}, T/\mathfrak{m}^j) \arrow[dll, out=0, in=180] & \\
& \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^i) \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^{i+j}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^j) \arrow[r] & 0
\end{tikzcd}$$

where the left top and bottom horizontal maps are given by the multiplication by π^k, where π is a generator of the maximal ideal. Note that the exactness of the last row follows from the definition of propagated Selmer structures.

However, it is important to note that the above exact sequence is only well defined because $\mathcal{F}$ is cartesian. Indeed, if we consider the following map given by Proposition 4.3.11:

$$\mathbf{H}^0(K_\mathfrak{u}, T/\mathfrak{m}^j) \to \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^i)$$

Then we need to show that the image is contained in $\mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^i)$. By Proposition 4.3.11, this is equivalent to

$$\ker\left([\pi^j]:\ \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^i) \to \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^{i+j})\right) \subset \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T/\mathfrak{m}^i)$$

Assume that

$$x \in \ker\left([\pi^j] : \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^i) \to \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^{i+j})\right)$$

Consider the diagram with exact rows induced by Proposition 4.3.11

$$\begin{tikzcd}
\mathbf{H}^1(K_\mathfrak{u}, T) \arrow[r, "\pi^i"] \arrow[d] & \mathbf{H}^1(K_\mathfrak{u}, T) \arrow[r, "\Pi"] \arrow[d, "\pi^j"] & \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^i) \arrow[r] \arrow[d, "{[\pi^j]}"] & \mathbf{H}^2(K_\mathfrak{u}, T)[\mathfrak{m}^i] \arrow[d, "\subset"] \\
\mathbf{H}^1(K_\mathfrak{u}, T) \arrow[r, "\pi^{i+j}"] & \mathbf{H}^1(K_\mathfrak{u}, T) \arrow[r] & \mathbf{H}^1(K_\mathfrak{u}, T/\mathfrak{m}^{i+j}) \arrow[r] & \mathbf{H}^2(K_\mathfrak{u}, T)[\mathfrak{m}^{i+j}]
\end{tikzcd}$$

Since the rightmost vertical map is injective, x belongs to kernel of the rightmost top horizontal map, so the exactness implies the existence of

$$y \in \mathbf{H}^1(K_{\mathfrak{u}}, T)$$

such that $\Pi(y) = x$. Since x is in the kernel of $[\pi^j]$, we can deduce that

$$\pi^j y \in \mathfrak{m}^{i+j}\mathbf{H}^1(K_{\mathfrak{u}}, T)$$

so, there is $z \in \mathbf{H}^1(K_{\mathfrak{u}}, T)$ such that $\pi^j y = \pi^{i+j} z$. Therefore,

$$\pi^j(y - \pi^i z) = 0 \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$$

Since $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T)$ is torsion-free by Definition 5.1.5, then $y - \pi^i z \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$. However,

$$\pi^i z \in \mathfrak{m}^i\mathbf{H}^1(K_{\mathfrak{u}}, T) = \ker(\Pi)$$

and Definition 2.2.7 thus implies that

$$x = \Pi(y - \pi^i z) \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^j)$$

proving that the exact sequence is well-defined.

The linearity on k implies that, for every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, there is some $\alpha_{\mathfrak{u}}$ the quantity

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)}{\#\mathbf{H}^0(K_{\mathfrak{u}}, T/\mathfrak{m}^k)} = \alpha_{\mathfrak{u}} \cdot \big(\#(R/\mathfrak{m})\big)^k \ \forall k \in \mathbb{N}$$

Combining that with (5.9), there is some $\alpha \in \mathbb{N}$ such that

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \alpha \cdot \big(\#(R/\mathfrak{m})\big)^k \ \forall k \in \mathbb{N} \tag{5.10}$$

By Propositions 5.1.17 and 5.3.3, the following quantity is a natural number bounded uniformly in k

$$\frac{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee/\mathfrak{m}^k}{\#\mathbf{H}^1_{\mathcal{F}}(K, T)/\mathfrak{m}^k} \frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])}$$

By Definition 2.2.14 and (5.10), we can conclude that

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \chi(\mathcal{F}) \cdot \big(\#(R/\mathfrak{m})\big)^k \quad \forall k \in \mathbb{N} \qquad \square$$

Remark 5.3.6. The core rank can be computed in terms of the local cohomology. If Assumption (BI) holds, then Remark 5.1.3 implies that

$$\mathbf{H}^0(K^{\mathfrak{n}}/K, T/\mathfrak{m}^k) = \mathbf{H}^0(K^{\mathfrak{n}}/K, T^*[\mathfrak{m}^k]) = 0 \ \forall k \in \mathbb{N}$$

Then Lemma 5.3.4 implies that

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])} = \prod_{\mathfrak{u}|\mathfrak{n}} \frac{\#\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)}{\#\mathbf{H}^0(K_{\mathfrak{u}}, T/\mathfrak{m}^k)}$$

Since the local conditions on T are closed submodules, they can be obtained as the inverse limit of their propagated local conditions, so we can then deduce that

$$\chi(\mathcal{F}) = \sum_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \big(\mathrm{rank}_R \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) - \mathrm{rank}_R \mathbf{H}^0(K_{\mathfrak{u}}, T)\big)$$

Proposition 5.3.5 will show that the core ranks of the propagated structures are $\chi(\mathcal{F})$ for all k, once we show that these structures are cartesian. That can be shown using a similar argument, which follows the one in [MR04, Lemma 3.7.1].

Proposition 5.3.7. ([MR04, Lemma 3.7.1]) Assume that $\mathcal{F}$ is a cartesian Selmer structure defined on T, as in Definition 5.1.5. For all $k \in \mathbb{N}$, the propagated Selmer structure in $T/\mathfrak{m}^k T$ is a cartesian Selmer structure, as in Definition 2.2.11.

Proof. For every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, we need to show that the map

$$[\pi^{k-1}] : \ \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}) \to \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)$$

is injective. Consider the following diagram with exacts rows from the proof of Proposition 5.3.5

$$\begin{tikzcd}
\mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[r, "\pi"] \arrow[d] & \mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[r, "\Pi_1"] \arrow[d, "\pi^{k-1}"] & \mathbf{H}^1(K_{\mathfrak{u}}, T/\mathfrak{m}) \arrow[r, "\delta_1"] \arrow[d, "{[\pi^{k-1}]}"] & \mathbf{H}^2(K_{\mathfrak{u}}, T)[\mathfrak{m}] \arrow[d, "\subset"] \\
\mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[r, "\pi^k"] & \mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[r, "\Pi_k"] & \mathbf{H}^1(K_{\mathfrak{u}}, T/\mathfrak{m}^k) \arrow[r, "\delta_k"] & \mathbf{H}^2(K_{\mathfrak{u}}, T)[\mathfrak{m}^k]
\end{tikzcd}$$

Let $c \in [\pi^{k-1}]^{-1}\big(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)\big)$. Since $[\pi^{k-1}](c) \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k)$, the definition of propagated local condition implies the existence of certain $d \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ such that $\Pi_k(d) = [\pi^{k-1}](c)$. Hence, $[\pi^{k-1}](c)$

is in the image of Π_k, which is also the kernel of δ_k. The injectivity of the rightmost vertical map implies that c is in the karnel of δ_1, so there exists $\alpha \in \mathbf{H}^1(K_{\mathfrak{u}}, T)$ such that $\Pi_1(\alpha) = c$. The exactness of the bottom row implies that

$$\pi^{k-1}\alpha - \gamma \in \ker(\Pi_k) = \pi^k \mathbf{H}^1(K_{\mathfrak{u}}, T)$$

Then there is $\beta \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ such that

$$\gamma = \pi^{k-1}\alpha - \pi^k\beta = \pi^{k-1}(\alpha - \pi\beta)$$

Since $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ is R-torsion free, then $\alpha - \pi\beta$ is also in $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$, so Definition 2.2.7 implies that

$$c = \Pi_1(\alpha) = \Pi_1(\alpha - \pi\beta) \in \Pi_1(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)) = \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m})$$

what concludes the proof of the injectivity of $[\pi^{k-1}]$. □

Proposition 5.3.8. Let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F}) \geq 0$. For all $k \in \mathbb{N}$, there is a non-canonical isomorphisms

$$\mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k) \cong (R/\mathfrak{m}^k)^{\chi(\mathcal{F})} \oplus \mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])$$

Proof. By the structure theorem, the proposition would follow from the identity

$$\frac{\#\mathbf{H}^1_{\mathcal{F}}(K, T)[\mathfrak{m}^k]}{\#\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)[\mathfrak{m}^k]} = \chi(\mathcal{F}) \cdot \big(\#(R/\mathfrak{m})\big)^k \ \forall k \in \mathbb{N}$$

What would follow from Proposition 5.3.5 and the identities

$$\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)[\mathfrak{m}^k] = \mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}^k])$$
$$\mathbf{H}^1_{\mathcal{F}}(K, T)[\mathfrak{m}^k] \cong \mathbf{H}^1_{\mathcal{F}}(K, T/\mathfrak{m}^k)$$

The first identity follows from Proposition 5.1.17. In order to show the second one, note that Remark 5.1.3 proves that the following map is injective for all $i \leq k$:

$$\mathbf{H}^1_{\mathcal{F}}(K^{\Sigma_{\mathcal{F}}}/K, T/\mathfrak{m}^i) \hookrightarrow \mathbf{H}^1_{\mathcal{F}}(K^{\Sigma_{\mathcal{F}}}/K, T/\mathfrak{m}^k)[\mathfrak{m}^i]$$

Then, for every $i \leq k$, the exact sequence

$$0 \longrightarrow T/\mathfrak{m}^i \longrightarrow T/\mathfrak{m}^k \longrightarrow T/\mathfrak{m}^{k-i}$$

induces a long exact sequence by Proposition 4.2.3 which, combined with Remark 5.1.3, produces an isomorphism

$$[\pi^{k-i}]: \ \mathbf{H}^1(K^{\Sigma_{\mathcal{F}}}/K, T/\mathfrak{m}^i) \to \mathbf{H}^1(K^{\Sigma_{\mathcal{F}}}/K, T/\mathfrak{m}^k)[\mathfrak{m}^i]$$

Consider the following diagram whose ro

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}\left(K, \frac{T}{\mathfrak{m}^i}\right) \arrow[r] \arrow[d, "{[\pi^{k-i}]}"] & \mathbf{H}^1_{\mathcal{F}}\left(K^{\Sigma_{\mathcal{F}}}/K, \frac{T}{\mathfrak{m}^i}\right) \arrow[r] \arrow[d, "{[\pi^{k-i}]}"] & \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \mathbf{H}^1_{/\mathcal{F}}\left(K_{\mathfrak{u}}, \dfrac{T}{\mathfrak{m}^i}\right) \arrow[d, "{[\pi^{k-i}]}"] \\
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}\left(K, \frac{T}{\mathfrak{m}^k}\right)[\mathfrak{m}^i] \arrow[r] & \mathbf{H}^1_{\mathcal{F}}\left(K^{\Sigma_{\mathcal{F}}}/K, \frac{T}{\mathfrak{m}^k}\right)[\mathfrak{m}^i] \arrow[r] & \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}}} \mathbf{H}^1_{/\mathcal{F}}\left(K_{\mathfrak{u}}, \dfrac{T}{\mathfrak{m}^k}\right)
\end{tikzcd}$$

We have already seen that the middle vertical map is an isomorphism and the rightmost vertical map is injective by Proposition 5.3.7. Hence, the snake lemma ensures that the leftmost vertical map is an isomorphism. □

We can generalise [MR04, Theorem 5.2.12] and [BSS25, Theorem 5.2] to compute the Fitting ideals of the Selmer group using a primitive ultra Kolyvagin system.

Theorem 5.3.9. Assume that $\chi(\mathcal{F}) > 0$ and let $\kappa \in \mathbf{KS}(T, \mathcal{F})$ be a primitive Kolyvagin system. For all $i \in \mathbb{Z}_{\geq 0}$, the following equality holds:

$$\Theta_i(\kappa) = \mathrm{Fitt}^i(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee)$$

Proof. Fix $i \in \mathbb{Z}_{\geq 0}$ and denote by $\overline{\kappa} = (\kappa^{(k)})_{k\in\mathbb{N}} \in \overline{\mathbf{KS}}(T, \mathcal{F})$ the image of κ under the isomorphism in Theorem 5.2.19. By Theorem 5.2.30, we now that

$$\mathbf{\Theta}_i(\kappa) = \mathbf{\Theta}_i(\overline{\kappa}) = \varprojlim_{k\in\mathbb{N}} \mathbf{\Theta}_i(\kappa^{(k)})$$

Similarly, Proposition 5.1.17 implies that

$$\mathrm{Fitt}^i(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee) = \varprojlim_{k\in\mathbb{N}} \mathrm{Fitt}^i(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*_k)^\vee)$$

Then the theorem follows from their analogues with finite coefficients (see [MR04, Theorem 5.2.12] and [BSS25, Theorem 5.2]). □

Remark 5.3.10. The fact that $\mathbf{KS}(T, \mathcal{F})$ is a free, cyclic R-module (shown in Corollary 5.2.20) and the relation between the theta ideals of primitive ultra Kolyvagin systems and the Fitting ideal of the Selmer group (see Theorem 5.3.9) can be proven without using their finite analogues. That can be achieved by developing a theory of ultra Kolyvagin systems over discrete valuation rings. The interested reader might find the theory in Chapter 6 useful, where this theory is developed when the coefficient ring is the Iwasawa algebra. The same arguments as in Theorems 6.3.5 and 6.3.19 (or even simplified versions) will work to give a description of the ultra Kolyvagin systems in the case where the coefficient ring is a discrete valuation ring.

5.4 Selmer groups of core rank 0

The goal of this section is a generalization of Theorems 3.3.7 and 3.4.1 to study Selmer groups with coefficients in discrete valuation ring coming from structures of core rank 0.

Assumption (CR0). In this section, we assume that we have given two patched cartesian Selmer structures $\mathcal{F} \leq \mathcal{G}$ of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$.

We start by proving an analogue of Proposition 3.3.3

Proposition 5.4.1. Let $\mathcal{F} \leq \mathcal{G}$ be as in Assumption (CR0). Then there is a unique ultraprime prime $\mathfrak{u}$ such that

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \subsetneq \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$$

For this ultraprime $\mathfrak{u}$, there is a non-canonical homomorphism

$$\mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) := \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) \big/ \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cong R$$

Proof. Since $\mathcal{F}$ is cartesian, the module

$$\bigoplus_{\mathfrak{u} \in \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)$$

is a finitely generated, free R-module. In addition, the global duality exact sequence in Proposition 2.2.6 and the definition of core ranks implies that its rank is one, which completes the proof of this proposition. $\square$

We can now generalise Definition 3.3.6 to the patched cohomology formalism.

Definition 5.4.2. Let $\kappa \in \mathrm{KS}(\mathcal{G})$ be an ultra Kolyvagin system and let $\mathfrak{u}$ the prime such that $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)$ is different to $\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$. For each $i \in \mathbb{Z}_{\geq 0}$, we can then define the ideals

$$\mathbf{\Theta}^{(0)}_i(\kappa, \mathcal{F}) := \sum_{\mathfrak{n} \in \mathcal{N}_i(T)} \mathrm{ind}\big(\mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}}), \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)\big)$$

Notation 5.4.3. We usually denote

$$\delta_n = \delta_n(\kappa, \mathcal{F}) := \mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}}) \in \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \cong R$$

Like Theorem 5.2.30, these theta ideals can be written as the limit of their finite analogues.

Proposition 5.4.4. Let $\kappa \in \mathrm{KS}(T, \mathcal{G})$ and let $\overline{\kappa} = \left(\kappa^{(k)}\right)_{k\in\mathbb{N}} \in \overline{\mathrm{KS}}(T, \mathcal{G})$ be its image under the isomorphism in Theorem 5.2.19. Then

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) = \varprojlim_{k\in\mathbb{N}} \Theta_i^{(0)}(\kappa^{(k)}, \mathcal{F})$$

where $\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F})$ is the ideal of R defined in Definition 5.4.2 and the ideals $\Theta_i^{(0)}(\kappa^{(k)}, \mathcal{F})$ were defined in Definition 3.3.6.

Proof. Recall that, by Propositions 5.3.7 and 5.3.8, the propagations of $\mathcal{F}$ and $\mathcal{G}$ to the finite quotients are cartesians and of core ranks 0 and 1, respectively, so Proposition 3.3.3 proves that the propagated local conditions only differ at one ultraprime[1], which has to be $\mathfrak{u}$. By construction, it is clear that

$$\mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^k) \cong \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)/\mathfrak{m}^k$$

Taken that into account, the proof is identical to Theorem 5.2.19, which will be sketched here. It is based in the following comparison of indices: for every $\mathfrak{n} \in \mathcal{N}(T)$, we can compute

$$\mathrm{ind}\big(\delta_{\mathfrak{n}}(\kappa, F)\big) + I_k = \mathrm{ind}\big(\delta_{\mathfrak{n}}(\kappa^{(k)}, \mathcal{F})\big)$$

which leads to the equality in the limit

$$\mathrm{ind}(\delta_{\mathfrak{n}}(\kappa, \mathcal{F})) = \varprojlim_{k\in\mathbb{N}} \delta_{\mathfrak{n}}(\kappa^{(k)}, \mathcal{F})$$

Similarly, if $\mathfrak{n}$ is represented by the sequence $(n_i)_{i\in\mathbb{N}}$ of square free product of Kolyvagin primes, then, for all $k \in \mathbb{N}$, we have that

$$\mathrm{ind}\big(\delta_{\mathfrak{n}}(\kappa^{(k)}, \mathcal{F})\big) = \mathrm{ind}\big(\delta_{n_i}(\kappa^{(k)}, \mathcal{F})\big)$$

for $\mathcal{U}$-many i. Taking these equalities into account, the argument in Theorem 5.2.30 deduces that

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) = \varprojlim_{k\in\mathbb{N}} \Theta_i^{(0)}(\kappa^{(k)}, \mathcal{F}) \qquad \square$$

[1]Although, in Proposition 3.3.3, the two Selmer structures differ at a unique prime or, equivalently, constant ultraprime, in this setting it is allowed that such ultraprime is non-constant since we are dealing with patched Selmer structures.

We can now show an analogue of Theorem 3.3.7.

Theorem 5.4.5. Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$ and let $\kappa \in \mathrm{KS}(T)$ be a Kolyvagin system. Then

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) \ \forall i \in \mathbb{Z}_{\geq 0}$$

In addition, if κ is primitive and, for some fixed index $i \in \mathbb{Z}_{\geq 0}$, one of the following conditions is satisfied:

- $i \leq \mathrm{rank}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$;
- $\mathbf{\Theta}_{i-1}^{(0)}(\kappa, \mathcal{F}) \subsetneq \mathrm{Fitt}_R^{i-1}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$;
- The Fitting ideal $\mathrm{Fitt}_R^{i-1}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$ does not vanish and there is a strict inclusion
$$\frac{\mathrm{Fitt}_R^{i-1}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}{\mathrm{Fitt}_R^{i}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)} \subsetneq \frac{\mathrm{Fitt}_R^{i}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}{\mathrm{Fitt}_R^{i+1}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)}$$
where the quotients are well-defined because all the Fitting ideals are non-zero and R is a principal artinian ring;

then the equality holds for the index i:

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Proof. By Proposition 5.1.17, we know that

$$\mathbf{H}^1_{\mathcal{F}^*}(K, (T/\mathfrak{m}^k)^*)^\vee = \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee/\mathfrak{m}^k$$

Hence,

$$\mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) + \mathfrak{m}^k = \mathrm{Fitt}^i_{R/\mathfrak{m}^k}\Big(\mathbf{H}^1_{\mathcal{F}^*}\big(K, (T/\mathfrak{m}^k)^*\big)^\vee\Big)$$

We can express this equality in the limit too:

$$\mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) = \varprojlim_{k \in \mathbb{N}} \mathrm{Fitt}^i_{R/\mathfrak{m}^k}\Big(\mathbf{H}^1_{\mathcal{F}}\big(K, (T/\mathfrak{m}^k)^*\big)^\vee\Big)$$

By Theorem 3.3.7 and Proposition 5.4.4, we can thus deduce that

$$\mathbf{\Theta}_i^{(0)}(\kappa, \mathcal{F}) \subset \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

We now show the equality holds under if one of the three assumptions holds:

- If $i \leq \operatorname{rank}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big) \leq \operatorname{rank}_{R/\mathfrak{m}^k}\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*_k)^\vee\big)$, then Theorem 3.3.7 implies that

$$\Theta_i^{(0)}(\kappa^{(k)}) = \operatorname{Fitt}^i_{R/\mathfrak{m}^k}\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)\ \forall k \in \mathbb{N}$$

  Taking limits, we can conclude that

$$\mathbf{\Theta}_i^{(0)}(\kappa,\mathcal{F}) = \operatorname{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$$

- Assume that $\mathbf{\Theta}_{i-1}^{(0)}(\kappa,\mathcal{F}) \subsetneq \operatorname{Fitt}^{i-1}_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$. We can assume, without loss of generality, that the Fitting ideal is non-zero. Hence, for large enough k, there is a strict inclusion

$$\Theta_{i-1}^{(0)}(\kappa^{(k)},\mathcal{F}) \subsetneq \operatorname{Fitt}^{i-1}_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)$$

  For those values of k, Theorem 3.3.7 implies that

$$\Theta_i^{(0)}(\kappa^{(k)},\mathcal{F}) = \operatorname{Fitt}^i_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)$$

  Then the equality also holds in the limit:

$$\mathbf{\Theta}_i^{(0)}(\kappa,\mathcal{F}) = \operatorname{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)$$

- Assume the third hypothesis holds, which implies that the Fitting ideal $\operatorname{Fitt}^{i-1}_R\big(H^1_{\mathcal{F}^*}(K,T^*)^\vee\big) = \mathfrak{m}^{k_0}$, for some $k_0 \in \mathbb{N}$. Whenever $k > k_0$, $\operatorname{Fitt}^i_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right) \neq 0$ and

$$\frac{\operatorname{Fitt}^{i-1}_R\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)}{\operatorname{Fitt}^i_R\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)} \subsetneq \frac{\operatorname{Fitt}^i_R\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)}{\operatorname{Fitt}^{i+1}_R\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)}$$

  Then Theorem 3.3.7 implies that

$$\Theta_i^{(0)}(\kappa^{(k)},\mathcal{F}) = \operatorname{Fitt}^i_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)\ \forall k \geq k_0$$

  Then, the equality also holds in the limit:

$$\Theta_i^{(0)}(\kappa^{(k)},\mathcal{F}) = \operatorname{Fitt}^i_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right) \qquad \square$$

Analogously to Theorem 3.4.1, Theorem 5.4.5 can be improved when T is not a self-dual Galois representation.

Theorem 5.4.6. Assume, in addition to the assumptions in Theorem 5.4.5 that T satisfies Assumption (NSD). Then

$$\mathbf{\Theta}_i^{(0)}(\kappa,\mathcal{F}) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)\ \forall i\in\mathbb{Z}_{\geq 0}$$

Proof. By Proposition 5.1.17, we know that

$$\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee = \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee/\mathfrak{m}^k\quad\forall k\in\mathbb{Z}_{\geq 0}$$

Similarly, Proposition 5.4.4 implies that

$$\mathbf{\Theta}_i^{(0)}(\kappa,\mathcal{F}) = \varprojlim_{k\in\mathbb{N}} \Theta_i^{(0)}(\kappa^{(k)},\mathcal{F})$$

It is clear that the representation $T/\mathfrak{m}^k$ satisfies Assumption (NSD) for every $k\in\mathbb{N}$. Hence, Theorem 3.4.1 implies that

$$\Theta_i^{(0)}(\kappa^{(k)},\mathcal{F}) = \mathrm{Fitt}^i_{R/\mathfrak{m}^k}\left(\mathbf{H}^1_{\mathcal{F}^*}\big(K,(T/\mathfrak{m}^k)^*\big)^\vee\right)\ \forall i\in\mathbb{Z}_{\geq 0},\ k\in\mathbb{N}$$

Taking limits, we obtain that

$$\mathbf{\Theta}_i^{(0)}(\kappa,\mathcal{F}) = \mathrm{Fitt}_R^i\big(\mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee\big)\ \forall i\in\mathbb{Z}_{\geq 0} \qquad \square$$

Remark 5.4.7. Theorem 5.4.6 is a generalization of a result of R. Sakamoto in his work [Sak24], who proved a similar computation of the Fitting ideals of $H^1_{\mathcal{F}^*}(K,T^*)^\vee$, using a different formalism of Kolyvagin systems of rank 0 (see [Sak21]) under the stronger assumption $H^1_{\mathcal{F}}(K,T)=0$.

We finish this section with a result that highlights that Theorem 5.4.5 is enough to determine the Selmer group up to isomorphism.

Corollary 5.4.8. Let $\mathcal{G}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{G})=1$ and let $\kappa\in\mathbf{KS}(T,\mathcal{F})$. Assume $\mathcal{F}\leq\mathcal{G}$ is another cartesian Selmer structure of core rank $\chi(\mathcal{F})=0$. Note that, for every $i\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$, we can find some $n_i\in\mathbb{Z}_{\geq 0}$ such that $\Theta_i^{(0)}(\widetilde{\kappa},\mathcal{F})=\mathfrak{m}^{n_i}$. Then all the Fitting ideals of the Selmer group can be computed as

$$\mathrm{Fitt}_R^i\left(H^1_{\mathcal{F}^*}(K,T^*)^\vee\right) = \mathfrak{m}^{\min\left\{n_i,\frac{n_{i+1}+n_{i-1}}{2}\right\}}\ \forall i\in\mathbb{Z}_{\geq 0}$$

Proof. Write $\mathrm{Fitt}^i_R\left(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\right) = \mathfrak{m}^{m_i}$ for some $m \in \mathbb{N} \cup \{0, \infty\}$. Then the inclusion in Theorem 5.4.5 implies that $n_i \geq m_i$. If $m_i = \infty$, then $n_i = m_i$, so we can assume, without loss of generality, that m_i is finite.

By the structure theorem of finitely generated modules over principal ideal domains and Proposition B.2.6, the following inequality holds for $i \in \mathbb{N}$:

$$m_i - m_{i+1} \leq m_{i-1} - m_i$$

Hence the index m_i can be upper bounded using m_{i-1} and m_{i+1}:

$$m_i \leq \frac{m_{i+1} + m_{i-1}}{2} \tag{5.11}$$

Assume that $n_i = m_i$. Then

$$m_i \leq \frac{m_{i+1} + m_{i-1}}{2} \leq \frac{n_{i+1} + n_{i-1}}{2} \Rightarrow m_i = \min\left\{n_i, \frac{n_{i+1} + n_{i-1}}{2}\right\}$$

Otherwise, assume that $n_i > m_i$. Then the third condition in Theorem 5.4.5 cannot hold, which means that the equality in (5.11) must hold.

The equality in Theorem 5.4.5 can be applied to the index $i + 1$. Since the second assumption in this theorem holds by our assumption, we obtain that $m_{i+1} = n_{i+1}$.

On the other hand, assume for the sake of contradiction that $n_{i-1} > m_{i-1}$. Then Theorem 5.4.5 would imply that $n_i = m_i$, contradicting our assumption. Therefore, $n_{i-1} = m_{i-1}$. We then have that

$$m_i = \frac{m_{i+1} + m_{i-1}}{2} = \frac{n_{i+1} + n_{i-1}}{2} = \min\left\{n_i, \frac{n_{i+1} + n_{i-1}}{2}\right\} \qquad \square$$

We finalise this section generalising Theorem 3.3.15. In order to do that, we define the theta ideal at infinity.

Definition 5.4.9. Let $\mathcal{F} \leq \mathcal{G}$ be Selmer structures of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$ and let $\kappa \in \mathbf{KS}(\mathcal{G})$ be an ultra Kolyvagin system. Define

$$\mathbf{\Theta}^{(0)}_\infty(\kappa, \mathcal{F}) = \sum_{i \geq 0} \mathbf{\Theta}^{(0)}_i(\kappa, \mathcal{F}) = \sum_{\mathfrak{n} \in \mathcal{N}} \mathrm{ind}\big(\mathrm{loc}_\mathfrak{u}(\kappa_\mathfrak{n}), \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_\mathfrak{u}, T)\big)$$

The following can be deduced from Theorem 5.4.5.

Theorem 5.4.10. Let $\mathcal{F} \leq \mathcal{G}$ be Selmer structures of core ranks $\chi(\mathcal{F}) = 0$ and $\chi(\mathcal{G}) = 1$ and let $\kappa \in \mathbf{KS}(\mathcal{G})$ be an ultra Kolyvagin system. Then κ is primitive if and only if $\mathbf{\Theta}^{(0)}_{\infty}(\kappa, \mathcal{F}) = R$ or, equivalently, if there exists some $\mathfrak{n} \in \mathcal{N}$ such that

$$\operatorname{ind}\big(\operatorname{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}}), \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)\big) = R$$

Proof. Assume κ is a primitive Kolyvagin system. Since $\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)$ is cofinitely generated, the ideals $\operatorname{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^{\vee}\big)$ are equal to R for large enough i. Using the second assumption of Theorem 5.4.5, we can find some $i \in \mathbb{Z}_{\geq 0}$ such that $\mathbf{\Theta}^{(0)}_{i}(\kappa, \mathcal{F}) = R$ (see Remark 3.3.9). Then, $\mathbf{\Theta}^{(0)}_{\infty}(\kappa, \mathcal{F}) = R$ as well.

Conversely, when κ is not primitive, there is some $\alpha \in \mathfrak{m}$ and $\widetilde{\kappa} \in \mathbf{KS}(\mathcal{G})$ such that $\kappa = \alpha\widetilde{\kappa}$. Then

$$\mathbf{\Theta}^{(0)}_{\infty}(\kappa, \mathcal{F}) = (\alpha)\mathbf{\Theta}^{(0)}_{\infty}(\widetilde{\kappa}, \mathcal{F}) \subset \mathfrak{m} \qquad \square$$

Chapter 6

Kolyvagin systems over the Iwasawa algebra

The theory of Kolyvagin systems developed in Chapter 3 described the Fitting ideals of Selmer groups when the coefficient ring is a principal ring. In this chapter, we generalise the theory to study Selmer groups whose coefficient ring is the Iwasawa algebra

$$\Lambda := \mathbb{Z}_p[[X]]$$

This chapter can be seen as a generalization of the theory previously developed in [BS21], [Sak18] and [BSS25]. Here, we adapt the construction in loc. cit., concerning Selmer groups with self-injective rings to construct a theory of Kolyvagin and Stark systems for Selmer groups with coefficients in the Iwasawa algebra.

The advantage of this new theory is that the Selmer group can be computed up to pseudo-isomorphism using the Kolyvagin derivatives of the Euler system, contrasting the description [BSS25] that used the theta ideals of the Stark system for this description.

Throughout this chapter, suppose that $\mathbf{T}$ be a free, finitely generated Iwasawa module endowed with a Galois action that is ramified only at finitely many primes.

Following the notation in Assumption (R-lim), we can express the Iwasawa algebra as a limit of its quotients. Define:

$$\Lambda_k := \Lambda/(p^k, X^k), \quad \mathbf{T}_k := \mathbf{T} \otimes_\Lambda \Lambda_k$$

Hence the Iwasawa algebra can be recovered as the inverse limit

$$\Lambda = \varprojlim_{k \in \mathbb{N}} \Lambda_k$$

where all Λ_n are self-injective rings. By Proposition 4.3.6, any patched Selmer group can be expressed as the limit

$$\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}) = \varprojlim_{k\in\mathbb{N}} \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}_k) = \varprojlim_{k\in\mathbb{N}} \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}/(p^k,X^k))$$

Since every Selmer group in the limit has a self-injective coefficient ring Λ_n, the work in [BSS25] constructed Kolyvagin and Stark systems for those Selmer groups (see Definitions 6.2.4 and 6.3.1).

If $\kappa^{(k)}$ and $\varepsilon^{(k)}$ are primitive Kolyvagin and Stark systems, respectively, for $\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}_n)$, [BSS25, Theorems 4.6 and 5.2] shows that

$$\mathbf{\Theta}\big(\kappa^{(k)}\big) \subset \mathbf{\Theta}\big(\varepsilon^{(k)}\big) = \mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*_k)^\vee\big) \tag{6.1}$$

By the limit argument in [BSS25, Theorem 4.12], we can use the Stark system to compute the Fitting ideals of the Iwasawa Selmer group:

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \varprojlim_{k\in\mathbb{N}} \mathbf{\Theta}\big(\varepsilon^{(k)}\big)$$

It is mentioned in [BSS25, Remark 5.3] that it is an interesting question to see whether the first inclusion in (6.1) is an equality. Here we do not approach this question in this generality, but we can show the equality holds for propagated Selmer structures from one defined on an Iwasawa Selmer group. However, the theory of ultra Kolyvagin systems allows us to treat Iwasawa modules directly, which are arguably the ultimate goal of this theory.

The innovative approach used in this chapter is the consideration of the patched cohomology groups defined in Chapter 4 to develop a theory of Kolyvagin systems over the Iwasawa algebra. This new approach allows us to use the structure theorem of finitely generated modules up to pseudo-isomorphism, relating the theta ideals constructed from both Kolyvagin and Stark systems to the Fitting ideals of the Selmer group.

The new theory presents some technical complications due to the lack of self-injectivity of the Iwasawa algebra. That is mainly solved using the results in §B.3.

In this section, we follow Definition 5.2.4 of Kolyvagin systems of higher rank (see also Definition 6.3.1 below) in which they are collections of elements

$$\kappa_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})$$

One can notice two main differences with respect to the original Definition 3.1.2. First, Selmer groups are contained in the patched cohomology and the classes forming Kolyvagin systems are in the exterior biduals of the Selmer group. This last modification deals with Selmer groups of core rank $\chi(\mathcal{F}) > 1$.

When the core rank is one, both definitions of Kolyvagin systems are equivalent. Indeed, suppose we are given a Kolyvagin system $\kappa = \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathcal{N}}$ as in Definition 3.1.2. There is a canonical map

$$\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})^{++} = \bigcap^1 \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T})$$

Considering the images of $\kappa_{\mathfrak{n}}$ under this map, we obtain a Kolyvagin system as in Definition 5.2.1.

From a Kolyvagin system, we define a sequence of theta ideals $\boldsymbol{\Theta}(\kappa)$ and prove they coincide up to finite index with the Fitting ideals of the Selmer group whenever the Kolyvagin system is primitive (see Notation 6.2.9 for the meaning of the symbol $\approx$):

$$\boldsymbol{\Theta}(\kappa) \approx \mathrm{Fitt}^i_{\Lambda}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^{\vee}\big)$$

If we impose one extra condition concerning the propagation of $\mathcal{F}$ to $\mathbf{T}/(X)$, the above can be shown to be an equality.

6.1 Iwasawa Selmer modules

In this chapter, we will assume that we are working with a Galois representation $\mathbf{T}$ over the Iwasawa algebra.

Assumption (Iw). Let $\Lambda = \mathbb{Z}_p[[X]]$ be the classical Iwasawa algebra and let $\mathbf{T}$ be a $\Lambda[[G_K]]$-module that is free and finitely generated as a Λ-module. In addition, we assume the Galois action is only ramified at finitely many primes.

The aim of this chapter is to develop a theory of Kolyvagin systems to study Selmer groups defined by cartesian structures on $\mathbf{T}$. In order to do that, we will work with the patched cohomology setting, as in Definition 4.3.1.

Notation 6.1.1. Following Assumption (R-lim), we denote

$$\Lambda_n = \Lambda/(p^n, X^n)$$

Since the maximal ideal is not principal, we cannot use Remark 5.1.10 to prove Assumption (Proj), which was required in Chapter

5. However, we can show that Assumption (Proj) is equivalent to the following.

Assumption (H2). There is a square-free product of ultraprimes $\mathfrak{n}$ divisible by all the primes in $\Sigma_{\mathcal{F}}$ such that the patched cohomology group $\mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})$ contains no finite Λ-submodules.

This assumption can be reinterpreted in terms of the local zero cohomology groups (see §6.1.2).

Proposition 6.1.2. Under Assumptions (BI), Assumptions (Proj) and (H2) are equivalent.

Proof. Fix a square-free product of ultraprimes $\mathfrak{n}$ divisible by all the primes in $\Sigma_{\mathcal{F}}$ and consider the exact sequence

$$0 \longrightarrow \mathbf{T} \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}/(X) \longrightarrow 0$$

By Remark 5.1.3 and Proposition 4.3.11, it induces another exact sequence

$$\mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T})/(X) \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/(X)) \twoheadrightarrow \mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})[X] \tag{6.2}$$

In addition, consider the exact sequence

$$0 \longrightarrow \mathbf{T}/(X) \xrightarrow{p} \mathbf{T}/(X) \longrightarrow \mathbf{T}/\mathfrak{m} \longrightarrow 0$$

Again, Remark 5.1.3 and Proposition 4.3.11 show that

$$\mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/(X))[p] = 0$$

and that the following map is injective:

$$\mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/(X))/(p) \hookrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/\mathfrak{m}) \tag{6.3}$$

The snake lemma applied to the short exact sequence in (6.2) and the multiplication by p induces another exact sequence

$$\mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})[\mathfrak{m}] \rightarrowtail \frac{\mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T})}{\mathfrak{m}} \longrightarrow \frac{\mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/(X))}{(p)} \tag{6.4}$$

Combining (6.3) and (6.4), we obtain an exact sequence

$$\mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})[\mathfrak{m}] \rightarrowtail \mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T})/\mathfrak{m} \longrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K, \mathbf{T}/\mathfrak{m})$$

that leads to the equivalence between Assumptions (Proj) and (H2) for the particular choice of $\mathfrak{n}$. $\square$

With Assumptions (BI), all the profinite Selmer groups are free modules.

Proposition 6.1.3. Let $\mathcal{F}$ be a Selmer structure. Under Assumptions (BI), the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K,T)$ is a free, finitely generated Λ-module.

Proof. By Proposition 5.1.4, $\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})$ is a finitely generated, torsion free Iwasawa module, so the structure theorem (see [NSW00, Theorem 5.3.8]) implies that it is pseudo-isomorphic to a free module. Since the Selmer group contains no finite submodules, there is a free module F and a finite module C fitting in the exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}) \longrightarrow F \longrightarrow C \longrightarrow 0$$

Assume for the sake of contradiction that C is a non-trivial module. The snake lemma induces another exact sequence

$$0 \longrightarrow C[X] \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})/(X) \longrightarrow F/(X) \longrightarrow C/(X)$$

When C is non-trivial, then $C[X] \neq 0$, so $\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})/(X)$ contains a finite submodule.

By the argument in Proposition 6.1.2, there is an injection

$$\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})/(X) \hookrightarrow \mathbf{H}^1_{\mathcal{F}}\big(K,\mathbf{T}/(X)\big)$$

what implies that the latter contains p-torsion. However, that contradicts Proposition 5.1.4 for the Galois representation $\mathbf{T}/(X)$. □

6.1.1 Core vertices

There are some special vertices in $\mathcal{N}$ that control the Kolyvagin and Stark systems defined below. In particular, Stark systems are controlled by weak core vertices and Kolyvagin systems by core vertices.

Definition 6.1.4. A *weak core vertex* is a square-free product $\mathfrak{c} \in \mathcal{N}$ of Kolyvagin ultraprimes such that $\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{c}}(K,\mathbf{T}^*) = 0$.

Proposition 6.1.5. If $\mathfrak{c} \in \mathcal{N}$ is a weak core vertex, then $\mathbf{H}^1_{\mathcal{F}^\mathfrak{c}}(K,\mathbf{T})$ is a free module of rank $\chi(\mathcal{F}) + \nu(\mathfrak{c})$.

Proof. By Proposition 5.1.37, $\chi(\mathcal{F}^c) = \chi(\mathcal{F}) + \nu(c)$, so the rank of $\mathbf{H}^1_{\mathcal{F}^\mathfrak{c}}(K,\mathbf{T})$ is $\chi(\mathcal{F}) + \nu(c)$. By Proposition 6.1.3,

$$\mathbf{H}^1_{\mathcal{F}^\mathfrak{c}}(K,\mathbf{T}) \cong \Lambda^{\chi(\mathcal{F})+\nu(\mathfrak{c})}$$ □

We can use Corollary 5.1.41 to prove the existence of a weak core vertex.

Proposition 6.1.6. Let $\mathfrak{n} \in \mathcal{N}$ be a square-free product of Kolyvagin ultraprimes. Then there exists a weak core vertex $\mathfrak{c} \in \mathcal{N}$ such that $\mathfrak{n} \mid \mathfrak{c}$.

Proof. It follows from a direct application of Corollary 5.1.41 to the Selmer structure $\mathcal{F}_n$. $\square$

Remark 6.1.7. Note that the following converse result also holds: when $\mathfrak{c} \in \mathcal{N}$ is a weak core vertex, every $\mathfrak{n} \in \mathcal{N}$ dividing $\mathfrak{c}$ is also a weak core vertex.

As we will show below in §6.2, weak core vertices are used to control Stark systems. In order to have an analogue for Kolyvagin systems, we need a stronger version of core vertices.

Definition 6.1.8. A *core vertex* is a square-free product $\mathfrak{c} \in \mathcal{N}$ of Kolyvagin ultraprimes such that $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{c})}(K, \mathbf{T}^*) = 0$.

Proposition 6.1.9. If $\mathfrak{c} \in \mathcal{N}$ is a core vertex, $\mathbf{H}^1_{\mathcal{F}(\mathfrak{c})}(K, \mathbf{T})$ is a free module of rank $\chi(\mathcal{F})$.

Proof. By Proposition 5.1.37, then $\chi(\mathcal{F}(\mathfrak{c})) = \chi(\mathcal{F})$, so the rank of $\mathbf{H}^1_{\mathcal{F}(\mathfrak{c})}(K, \mathbf{T})$ is $\chi(\mathcal{F})$. By Proposition 6.1.3, we have that

$$\mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T}) \cong \Lambda^{\chi(\mathcal{F})}$$

$\square$

We can also show the existence of core vertices, using Corollary 5.1.43.

Proposition 6.1.10. Let $\mathfrak{n} \in \mathcal{N}$ be a square-free product of Kolyvagin ultraprimes. Then there is a core vertex $\mathfrak{c} \in \mathcal{N}$ such that $\mathfrak{n} \mid \mathfrak{c}$.

Proof. It is an immediate consequence of Corollary 5.1.43. $\square$

Remark 6.1.11. There is no analogue of Remark 6.1.7 for core vertices, since there might exist $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ such that $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)$ vanishes but $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}\mathfrak{u})}(K, \mathbf{T}^*)$ does not.

6.1.2 Assumption on the second cohomology

In this section, we reinterpret Assumption (H2) on the lack of finite Λ-submodules of $\mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})$ in terms of the local cohomology of bad primes.

Assumption (Tam). For every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, $\mathbf{H}^0(K_{\mathfrak{u}}, \mathbf{T}^*)^\vee$ contains no finite Λ-submodule.

Remark 6.1.12. By Proposition 4.2.18, Assumption (Tam) is equivalent to $\mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T})$ not containing any finite submodules for any $\mathfrak{u} \in \Sigma_{\mathcal{F}}$.

Lemma 6.1.13. Under Assumption (BI), Assumption (Tam) implies Assumption (H2).

Proof. Let $\mathfrak{n}$ be a weak core vertex for $\mathcal{F}$, which exists by Proposition 6.1.6, and let Σ be the set consisting of the primes in $\Sigma_{\mathcal{F}}$ and the prime divisors of $\mathfrak{n}$. By Proposition 4.2.29, there is an exact sequence

$$\begin{tikzcd} \bigoplus_{\mathfrak{u} \in \Sigma} \mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}) \arrow[r] & \mathbf{H}^1(K^{\Sigma}/K, \mathbf{T}^*)^\vee \arrow[dl] \\ \mathbf{H}^2(K^{\Sigma}/K, \mathbf{T}) \arrow[r] & \bigoplus_{\mathfrak{u} \in \Sigma} \mathbf{H}^0(K_{\mathfrak{u}}, \mathbf{T}^*)^\vee \end{tikzcd}$$

Since $\mathfrak{n}$ is a weak core vertex, the first map is surjective, so the proof of the lemma is reduced to showing that $\mathbf{H}^0(K_{\mathfrak{u}}, \mathbf{T}^*)^\vee$ contains no finite non-trivial submodules for any $\mathfrak{u} \in \Sigma$. When $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, this is an assumption, and when $\mathfrak{u} \mid \mathfrak{n}$, then $\mathfrak{u}$ is a Kolyvagin prime, so

$$\mathbf{H}^0(K_{\mathfrak{u}}, \mathbf{T}^*)^\vee \cong \Lambda$$

also contains no finite non-zero submodules. Then Assumption (H2) holds by considering the square free product of primes in Σ. $\square$

Remark 6.1.14. When ℓ is a constant ultraprime, one condition implying that $\mathbf{H}^0(K_\ell, \mathbf{T}^*)^\vee$ contains no-finite submodules is the p-divisibility of the dual group $\mathbf{H}^0(K_\ell, \mathbf{T}^*)$.

Suppose that $\mathbf{T}$ comes from a p-adic representation T by Shapiro's lemma, meaning that

$$\mathbf{T} = T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$$

where K_∞/K is a $\mathbb{Z}_p$-extension at which no-finite prime splits completely. Then Shapiro's lemma gives an isomorphism

$$\mathbf{H}^0(K_\ell, \mathbf{T}^*) = \bigoplus_{v|\ell} \mathbf{H}^0\big((K_\infty)_v, T^*\big)$$

where the sum is taken over all the primes of K_∞ above ℓ.

We assume the p-primality of the Tamagawa numbers associated with a Galois representation, which can be seen as the divisibility of the group $\mathbf{H}^0\big(K_\ell^{\mathrm{ur}}, T^*\big)$.

Note that $(K_\infty)_v/K_\ell$ is the unramified $\mathbb{Z}_p$-extension of K_ℓ. Hence the profinite degree $K_\ell^{\mathrm{ur}}/(K_\infty)_v$ is prime to p, so its action on

$$\mathrm{Aut}\Big(\mathbf{H}^0\big(K_\ell^{\mathrm{ur}}, T^*\big)^\vee\Big) \cong \mathrm{GL}_d(\mathbb{Z}_p)$$

where $d \in \mathbb{Z}_{\geq 0}$ is at most the rank of $\mathbf{T}$, has finite image. Hence the eigenvalues of all the matrices in the image are roots of unity, impliying that $\mathbf{H}^0\big((K_\infty)_v, T^*\big)$ is also a divisible group.

6.1.3 The cartesian condition

The hypotheses on the local cohomology groups (Tam) can be used to show an equivalence between the propagated structure on $\mathbf{T}/(X)$ being cartesian and the freeness of the quotients $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})$.

Proposition 6.1.15. Let $\mathcal{F}$ be a cartesian Selmer structure and assume that the propagated structure to $\mathbf{T}/(X)$ is cartesian too. Then, for every ultraprime $\mathfrak{u} \in \Sigma_\mathcal{F}$, $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})$ is a free Iwasawa module. In addition, the converse is true under Assumption (Tam).

Proof. Since $\mathcal{F}$ is cartesian, $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})$ is pseudo-isomorphic to a free module F. Since it does not contain any finite submodules, there is a finite module C fitting in the exact sequence

$$0 \longrightarrow \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}) \longrightarrow F \longrightarrow C \longrightarrow 0$$

When C is not the trivial module, then the snake lemma implies that $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})/(X)$ contains a finite submodule.

However, Proposition 4.2.6 and diagram chasing induce another exact sequence

$$\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})/(X) \rightarrowtail \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/(X)) \longrightarrow \mathbf{H}^2(K_\mathfrak{u}, \mathbf{T})[X]$$

When the propagated Selmer structure is cartesian, the quotient $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/(X))$ contains no finite Selmer modules, so neither does $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T})/(X)$. Therefore, $C = 0$ and $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T})$ is a free Iwasawa module.

Conversely, Remark 6.1.12 implies that, under Assumption (Tam), $\mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T})[X]$ contains no finite submodules. If $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T})$ is a free module, then $\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/(X))$ contains no finite submodules either, so the propagated Selmer group is cartesian. □

These conditions can be used to prove that the propagated Selmer structures to $\mathbf{T}_k$ are cartesian.

Proposition 6.1.16. Let $\mathcal{F}$ be a cartesian Selmer structure on $\mathbf{T}$ such that its propagation to $\mathbf{T}/(X)$ is also cartesian. In addition, suppose that Assumption (Tam) holds. Then the propagation of $\mathcal{F}$ to $\mathbf{T}_k$ is cartesian (see Definition 2.2.11) for all $k \in \mathbb{N}$.

Proof. We need to show, for every $\mathfrak{u} \in \Sigma_{\mathcal{F}}$, that the composition

$$\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/\mathfrak{m}) \xrightarrow{[p^{k-1}]} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/(p^k, X)) \xrightarrow{[X^{k-1}]} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}_k)$$

is injective. The first map $[p^{k-1}]$ is known to be injective by Proposition 5.3.7, since the propagated Selmer structure on $\mathbf{T}/(X)$ is cartesian and $\Lambda/(X)$ is a discrete valuation ring. For the injectivity of $[X^{k-1}]$, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/p^k) \arrow[r, "\Pi_1"] \arrow[d, "X^{k-1}"] & \mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/(p^k, X)) \arrow[r, "\delta_1"] \arrow[d, "{[X^{k-1}]}"] & \mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T}/p^k)[X] \arrow[d, "\subset"] \\
\mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/p^k) \arrow[r, "\Pi_k"] & \mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/(p^k, X^k)) \arrow[r, "\delta_k"] & \mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T}/p^k)[X^k]
\end{tikzcd}$$

Let $x \in [X^{k-1}]^{-1}\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/(p^k, X^k))$. By Definition 2.2.7, there is some $\gamma \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/p^k)$ such that $\Pi_k(\gamma) = [X^{k-1}](x)$. The exactness of the last row implies that $[X^{k-1}](x) \in \ker(\delta_k)$, so the injectivity of the rightmost vertical map implies that $x \in \ker(\delta_1) = \mathrm{Im}(\Pi_1)$.

Let $\alpha \in \mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/p^k)$ be such that $\Pi_1(\alpha) = x$. By Proposition 4.2.6, $X^k\mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/p^k) = \ker(\Pi_k)$, so we can find an element $\beta \in \mathbf{H}^1(K_{\mathfrak{u}}, \mathbf{T}/p^k)$ such that

$$\gamma = X^{k-1}\alpha + X^n\beta = X^{k-1}(\alpha + X\beta) \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/p^k)$$

We claim that $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k)$ contains no X-torsion. Indeed, there is an exact sequence

$$0 \longrightarrow \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})/p^k \longrightarrow \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k) \longrightarrow \mathbf{H}^2(K_\mathfrak{u}, \mathbf{T})[p^k]$$

By Proposition 6.1.15, $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})$ is a free module. Therefore, $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T})/p^k$ contains no X-torsion. In addition $\mathbf{H}^2(K_\mathfrak{u}, \mathbf{T})[p^k]$ cannot contain X-torsion either since that would contradict Assumption (Tam) (see also Remark 6.1.12). Hence $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k)$ cannot contain X torsion either.

Since $\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k)$ contains no X-torsion, then

$$\alpha + X\beta \in \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k)$$

Since $X\mathbf{H}^1(K_\mathfrak{u}, \mathbf{T}/p^k) = \ker(\Pi_1)$ by Proposition 4.2.6, we have that

$$x = \Pi_1(\alpha + X\beta) \in \Pi_1\big(\mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/p^k)\big) = \mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, \mathbf{T}/(p^k, X))$$

which proves that the propagated condition is cartesian. $\square$

Next we show that the core ranks of the propagated Selmer structures coincides with $\chi(\mathcal{F})$.

Proposition 6.1.17. Let $\mathcal{F}$ be a cartesian Selmer structure. The core rank of the propagated Selmer structure to $\mathbf{T}/(X)$ coincides with $\chi(\mathcal{F})$.

The previous result can be generalised to the propagated Selmer structures to the finite Selmer groups of Λ_k.

Corollary 6.1.18. Let $\mathcal{F}$ be a cartesian Selmer structure whose projection to $\mathbf{T}/(X)$ is also cartesian. For each $k \in \mathbb{N}$, the core rank of the propagated Selmer structure to Λ_k is $\chi(\mathcal{F})$.

Proof. By Definition 2.2.14, we need to show that

$$\chi(\mathcal{F}) = \dim_k \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}/\mathfrak{m}) - \dim_k \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*[\mathfrak{m}])$$

Since $\Lambda/(X)$ is a discrete valuation ring, the equality follows from Propositions 5.3.8 and 6.1.17. $\square$

Proof of Proposition 6.1.17. We define C as the cokernel fitting in the exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T})/(X) \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}/(X)) \longrightarrow C \longrightarrow 0$$

Since $\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})$ is pseudo-isomorphic to a free module, we have that

$$\mathrm{rank}_{\mathbb{Z}_p} C = \mathrm{rank}_{\mathbb{Z}_p} \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}/(X)) - \mathrm{rank}_{\Lambda} \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})$$

This proof will be based on the computation of the rank of C. By Corollary 5.1.41, we can find a square-free product of ultraprimes $\mathfrak{n}$, divisible by all the primes in $\Sigma_{\mathcal{F}}$, such that $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K,\mathbf{T}^*) = 0$. Define I as the image

$$I := \mathrm{Im}\left(\mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T}) \to \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},\mathbf{T})\right)$$

Then there is a short exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}) \longrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T}) \longrightarrow I \longrightarrow 0$$

Since $\mathcal{F}$ is cartesian, I is a torsion-free module, so the snake lemma induces another exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})/(X) \longrightarrow \mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T})/(X) \longrightarrow I/(X) \longrightarrow 0$$

Similarly, define J as the image

$$J := \mathrm{Im}\left(\mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T}/(X)) \to \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}},\mathbf{T}/(X))\right)$$

Then there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})/(X) \arrow[r] \arrow[d] & \mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T})/(X) \arrow[r] \arrow[d] & I/(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}/(X)) \arrow[r] & \mathbf{H}^1(K^{\mathfrak{n}}/K,\mathbf{T}/(X)) \arrow[r] & J \arrow[r] & 0
\end{tikzcd}$$

Note that Proposition 4.2.6 proves that the middle vertical map is injective with cokernel being isomorphic to $\mathbf{H}^2(K^{\mathfrak{n}}/K,\mathbf{T})[X]$. If we define A and B as the kernel and cokernel fitting in the exact sequence

$$0 \longrightarrow A \longrightarrow I/(X) \longrightarrow J \longrightarrow B \longrightarrow 0$$

the snake lemma induces an exact sequence

$$0 \longrightarrow A \longrightarrow C \longrightarrow \mathbf{H}^2(K^{\mathfrak{n}}/K,\mathbf{T})[X] \longrightarrow B \longrightarrow 0$$

Since the rank is an additive function, we have that

$$\mathrm{rank}_{\mathbb{Z}_p} C = \mathrm{rank}_{\mathbb{Z}_p} \mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})[X] + \mathrm{rank}_{\mathbb{Z}_p} A - \mathrm{rank}_{\mathbb{Z}_p} B$$

Our choice of $\mathfrak{n}$ and the combination of Proposition 2.2.6 and 5.1.17 produce a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & A \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & I/(X) \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T})/(X) \arrow[r] \arrow[d] & \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee/(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T}/(X)) \arrow[r] \arrow[d] & \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*[X])^\vee \arrow[r] \arrow[d] & 0 \\
 & B & \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T})[X] & 0 &
\end{tikzcd}$$

Note that the exactness of the middle column follows from Proposition 4.2.6 and the exactness of the rightmost column follows from Proposition 5.1.17. Combining the snake lemma with Proposition 4.2.29, we can construct isomorphisms

$$B = \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^2(K_{\mathfrak{u}}, \mathbf{T})[X] = \mathbf{H}^2(K^{\mathfrak{n}}/K, \mathbf{T})[X]$$

where the last isomorphism holds because $\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, \mathbf{T}^*) = 0$. Hence the ranks of A and C coincide. A double application of the snake lemma shows that

$$A = \ker \left(I/(X) \to \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, \mathbf{T})/(X) \right) = \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee[X]$$

By the structure theorem of finitely generated Iwasawa modules, we have that

$$\mathrm{rank}_{\mathbb{Z}_p} \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee[X] = \mathrm{rank}_{\mathbb{Z}_p} \frac{\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee}{(X)} - \mathrm{rank}_\Lambda \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee$$

By Proposition 5.1.17, we can then conclude that

$$\mathrm{rank}_{\mathbb{Z}_p} \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}/(X)) - \mathrm{rank}_{\mathbb{Z}_p} \mathbf{H}^1_{\mathcal{F}^*}(K, (\mathbf{T}/(X))^*)^\vee = \mathrm{rank}_\Lambda \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}) - \mathrm{rank}_\Lambda \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee$$

so the core ranks coincide. $\square$

6.2 Stark Systems

Let $\mathfrak{n}, \mathfrak{r} \in \mathcal{N}$ be square-free products of Kolyvagin ultraprimes such that $\mathfrak{n} \mid \mathfrak{r}$. There is an exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathcal{F}^{\mathfrak{r}}}(K, \mathbf{T}) \longrightarrow \prod_{\mathfrak{u} \mid \frac{\mathfrak{r}}{\mathfrak{n}}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T})$$

We want to use Corollary B.4.9 to construct a map

$$\phi_{\mathfrak{r},\mathfrak{n}} : \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{r})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{r}}}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{n})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T})$$

In order to do that, we fix, for every Kolyvagin ultraprime $\mathfrak{u} \in \mathcal{P}$, an isomorphism

$$\psi_{\mathfrak{u}} : \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T}) \cong \Lambda$$

In addition, we need to fix an order in the set $\mathcal{P}$. Under these choices, we have fixed an isomorphism

$$\prod_{\mathfrak{u} \mid \frac{\mathfrak{r}}{\mathfrak{n}}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T}) \cong \Lambda^{\nu(\mathfrak{r})-\nu(\mathfrak{n})}$$

Then the map $\phi_{\mathfrak{r},\mathfrak{n}}$ is the one given by Corollary B.4.9.

In order to guarantee the commutative property of these maps, we need to modify the sign convention. Following [BS21], we define

Definition 6.2.1. For every elements $\mathfrak{n}, \mathfrak{r} \in \mathcal{N}$ such that $\mathfrak{n} \mid \mathfrak{r}$, denote by $\mathfrak{u}_1, \ldots, \mathfrak{u}_{\nu(\mathfrak{n})}$ and $\mathfrak{u}'_1, \ldots, \mathfrak{u}'_{\nu(\mathfrak{r}/\mathfrak{n})}$ the prime divisors of $\mathfrak{n}$ and $\mathfrak{r}/\mathfrak{n}$, ordered by the fixed ordering. Similarly, let $\mathfrak{u}''_1, \ldots, \mathfrak{u}''_{\nu(\mathfrak{r})}$ the ordered set of prime divisors of $\mathfrak{r}$. Define $\mathrm{sign}(\mathfrak{r}, \mathfrak{n})$ as the sign of the permutation

$$(\mathfrak{u}'_1, \ldots, \mathfrak{u}'_{\nu(\mathfrak{r}/\mathfrak{n})}, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\nu(\mathfrak{n})}) \mapsto (\mathfrak{u}''_1, \ldots, \mathfrak{u}''_{\nu(\mathfrak{r})})$$

Definition 6.2.2. We define the map

$$\Phi_{\mathfrak{r},\mathfrak{n}} = \mathrm{sign}(\mathfrak{r}, \mathfrak{n}) \cdot \phi_{\mathfrak{r},\mathfrak{n}} : \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{r})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{r}}}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{n})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T})$$

where $\mathrm{sign}(\mathfrak{r}, \mathfrak{n})$ is defined in Definition 6.2.1 and the map $\phi_{\mathfrak{r},\mathfrak{n}}$ is the map given in Corollary B.4.9.

Lemma 6.2.3. Let $\mathfrak{n}, \mathfrak{r}, \mathfrak{s} \in \mathcal{N}$ be square-free products of Kolyvagin ultraprimes such that $\mathfrak{n} \mid \mathfrak{r} \mid \mathfrak{s}$. Then

$$\Phi_{\mathfrak{s},\mathfrak{n}} = \Phi_{\mathfrak{r},\mathfrak{n}} \circ \Phi_{\mathfrak{s},\mathfrak{r}}$$

Proof. By Proposition B.4.11, the map

$$\mathrm{sign}(\mathfrak{s},\mathfrak{r})\cdot\mathrm{sign}(\mathfrak{r},\mathfrak{n})\cdot(\Phi_{\mathfrak{s},\mathfrak{r}}\circ\Phi_{\mathfrak{r},\mathfrak{n}})$$

is the map induced from Corollary B.4.9 by the exact sequence

$$\mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K,\mathbf{T})\rightarrowtail\mathbf{H}^1_{\mathcal{F}^{\mathfrak{s}}}(K,\mathbf{T})\longrightarrow\prod_{\mathfrak{u}|\frac{\mathfrak{r}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\times\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{r}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})$$

and the isomorphisms

$$\prod_{\mathfrak{u}|\frac{\mathfrak{r}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\cong\Lambda^{\nu(\mathfrak{r}/\mathfrak{n})},\quad\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{r}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\cong\Lambda^{\nu(\mathfrak{s}/\mathfrak{r})}$$

induced by the order choice.

Similarly, the map $\mathrm{sign}(\mathfrak{s},\mathfrak{n})\Phi_{\mathfrak{s},\mathfrak{n}}$ is induced by the exact sequence

$$0\longrightarrow\mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K,\mathbf{T})\longrightarrow\mathbf{H}^1_{\mathcal{F}^{\mathfrak{s}}}(K,\mathbf{T})\longrightarrow\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})$$

and the isomorphism

$$\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},T)\cong\Lambda^{\nu(\mathfrak{s}/\mathfrak{n})}$$

Consider the following diagram in which the vertical isomorphisms are induced by the ones above mentioned and the horizontal ones are the natural maps.

$$\begin{tikzcd}
\det\left(\prod_{\mathfrak{u}|\frac{\mathfrak{r}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\times\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{r}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\right) \arrow[r] \arrow[d] & \det\left(\prod_{\mathfrak{u}|\frac{\mathfrak{s}}{\mathfrak{n}}}\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T})\right) \arrow[d] \\
\det\left(R^{\nu(\mathfrak{r}/\mathfrak{n})}\times R^{\nu(\mathfrak{s}/\mathfrak{r})}\right) \arrow[r] & \det\left(R^{\nu(\mathfrak{s}/\mathfrak{n})}\right)
\end{tikzcd}$$

Then this diagram is commutative up to the sign

$$\mathrm{sign}(\mathfrak{s},\mathfrak{r})\mathrm{sign}(\mathfrak{s},\mathfrak{n})\mathrm{sign}(\mathfrak{r},\mathfrak{n})$$

By the identifications made above, we get that

$$\Phi_{\mathfrak{s},\mathfrak{n}}=\Phi_{\mathfrak{s},\mathfrak{r}}\circ\Phi_{\mathfrak{r},\mathfrak{n}}$$ □

Therefore, the set of maps $\Phi_{\mathfrak{r},\mathfrak{n}}$ forms an inverse system, so it makes sense to consider the elements in the inverse limit.

Definition 6.2.4. The module of Stark systems of $\mathcal{F}$ is defined as the inverse limit

$$\mathbf{SS}(\mathcal{F}) := \varprojlim_{\mathfrak{n}\in\mathcal{N}} \left(\bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \right)$$

Analogously to the theory of Kolyvagin systems, the link between Stark systems and the Fitting ideals of Selmer groups involves the construction of a sequence of theta ideals.

Definition 6.2.5. Let $\varepsilon \in \mathbf{SS}(\mathcal{F})$ be a Stark system. We define its theta ideals as

$$\mathbf{\Theta}(\varepsilon) := \sum_{n\in\mathcal{N}_i} \operatorname{ind}\left(\varepsilon_n, \bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \right)$$

6.2.1 The module of Stark systems

The goal of this section is to prove the following theorem, which says that Stark systems are controlled by weak core vertices.

Theorem 6.2.6. Let $\mathfrak{c} \in \mathcal{N}$ be a weak core vertex. Then the projection map

$$\mathbf{SS}(\mathcal{F}) \to \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T})$$

is an isomorphism.

Proof. By Proposition 6.1.6, we only need to prove that if $\mathfrak{c} \in \mathcal{N}$ is a weak core vertex and $\mathfrak{u} \in \mathcal{P}$ does not divide $\mathfrak{c}$, the map

$$\bigcap^{r+\nu(\mathfrak{c}\mathfrak{u})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}\mathfrak{u}}}(K, \mathbf{T}) \to \bigcap^{r+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T})$$

is an isomorphism. This map is induced by the exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}\mathfrak{u}}}(K, \mathbf{T}) \longrightarrow \mathbf{H}^1_{s}(K_{\mathfrak{u}}, \mathbf{T}) \longrightarrow 0$$

Since $\mathrm{Ext}^1(\Lambda,\Lambda)=0$, the dual map $\mathbf{H}^1_{\mathcal{F}^{\mathfrak{cu}}}(K,\mathbf{T})^+ \to \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T})^+$ is surjective. Hence the following map is also surjective:

$$\bigwedge^{r+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T})^+ \to \bigwedge^{r+\nu(\mathfrak{cu})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{cu}}}(K,\mathbf{T})^+$$

and turns out to be an isomorphism since both are free Λ-modules of rank 1, by Proposition 6.1.9 and Lemma B.3.2. Therefore, its dual map is also an isomorphism. □

The above control theorem allows the computation of the module of Stark systems.

Corollary 6.2.7. The module of Stark systems $\mathbf{SS}(\mathcal{F})$ is a free Λ-module of rank one.

Proof. Let $\mathfrak{c}\in\mathcal{N}$ be a weak core vertex. By Proposition 6.1.9, $\mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T})$ is pseudo-isomorphic to $\Lambda^{\chi(\mathcal{F})+\nu(\mathfrak{c})}$. Thus, Lemma B.3.2 and Theorem 6.2.6 imply that

$$\mathbf{SS}(\mathcal{F}) \cong \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T}) \cong \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \Lambda^{\chi(\mathcal{F})+\nu(\mathfrak{c})} = \Lambda \qquad \square$$

Therefore, it makes sense to talk about generators of the module of Stark systems.

Definition 6.2.8. A Stark system is called *primitive* if it is a generator of $\mathbf{SS}(\mathcal{F})$.

The different elements of a primitive Stark system carry information about restricted Selmer groups. In order to establish this relation explicitly, we need to use the following notation.

Notation 6.2.9. Let M be an Iwasawa module and let I and J be two submodules. We say that I and J are pseudo-equal if, for every height one prime ideal β, then $I_\beta = J_\beta$ in M_β. We then denote $I \approx J$. Note that this definition also applies to two ideals I and J of the Iwasawa algebra. If, in addition, $I \supset J$, we will write $I \gtrapprox J$.

Theorem 6.2.10. Let $\varepsilon = (\varepsilon_{\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}} \in \mathbf{SS}(\mathcal{F})$ be a primitive Stark system. For every $\mathfrak{n}\in\mathcal{N}$, we have that $\mathrm{ind}(\varepsilon_n)$ is a principal ideal such that

$$\mathrm{ind}(\varepsilon_{\mathfrak{n}}) \gtrapprox \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K,\mathbf{T}^*)^\vee\big)$$

Proof. Fix $\mathfrak{n} \in \mathcal{N}$. By Remark 6.1.7, there exists a core vertex $\mathfrak{c}$ such that $\mathfrak{n} \mid \mathfrak{c}$. Consider then the map in Definition 6.2.2

$$\Phi_{\mathfrak{c},\mathfrak{n}} : \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{n})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T})$$

Since ε generates $\mathbf{SS}(\mathcal{F})$, Theorem 6.2.6 implies that $\varepsilon_{\mathfrak{c}}$ generates $\bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T})$. Since $\varepsilon_{\mathfrak{n}} = \Phi_{\mathfrak{c},\mathfrak{n}}(\varepsilon_{\mathfrak{c}})$, then Proposition B.4.14 implies that

$$\mathrm{ind}(\varepsilon_{\mathfrak{n}}) \approx \mathrm{Fitt}^0_{\Lambda}\big(\mathbf{H}^1_{\mathcal{F}_{\mathfrak{n}}}(K, \mathbf{T}^*)^{\vee}\big) \qquad \square$$

In addition, Proposition B.4.15 implies that

$$\mathrm{Fitt}^0_{\Lambda}\big(\mathbf{H}^1_{\mathcal{F}^*_{\mathfrak{n}}}(K, \mathbf{T}^*)^{\vee}\big) \subset \mathrm{ind}(\varepsilon_{\mathfrak{n}})$$

6.2.2 Higher Fitting ideals of the Selmer group

We can also use Stark systems to compute higher Fitting ideals of the Selmer group. Indeed, we will see the higher Fitting ideals coincide up to finite index with the theta ideals defined in Definition 6.2.5

Theorem 6.2.11. Let $\varepsilon \in \mathbf{SS}(\mathcal{F})$ be a primitive Stark system. Then

$$\boldsymbol{\Theta}(\varepsilon) \gtrapprox \mathrm{Fitt}^i_{\Lambda}\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^{\vee}\big)$$

Proof. Choose a weak core vertex $\mathfrak{c} \in \mathcal{N}$, which induces an exact sequence

$$0 \to \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}) \to \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T}) \to \bigoplus_{\mathfrak{u}\mid\mathfrak{c}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T}) \to \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^{\vee} \to 0$$

For every $\mathfrak{q} \mid \mathfrak{c}$, there is another exact sequence

$$0 \to \mathbf{H}^1_{\mathcal{F}^{\mathfrak{q}}}(K, \mathbf{T}) \to \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K, \mathbf{T}) \to \bigoplus_{\mathfrak{u}\mid\frac{\mathfrak{c}}{\mathfrak{q}}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T}) \to \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{q}}}(K, \mathbf{T}^*)^{\vee} \to 0$$

By the definition of the Fitting ideals, we can deduce that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}\mid\mathfrak{c}} \mathrm{Fitt}^{i-1}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{q}}(K,\mathbf{T}^*)^\vee\big)$$

Proposition 6.1.6 implies that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}\in\mathcal{P}} \mathrm{Fitt}^{i-1}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{q}}(K,\mathbf{T}^*)^\vee\big)$$

We can now proceed to prove Theorem 6.2.11 by induction. For $i = 0$, it is Theorem 6.2.10 for $n = 1$. Assuming the result holds true for $i - 1$, we can apply it to all Selmer structures $\mathcal{F}^\mathfrak{q}$ to obtain that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}\in\mathcal{P}} \sum_{\mathfrak{n}\in\mathcal{N}_{i-1},\ \mathfrak{q}\nmid\mathfrak{n}} \mathrm{Fitt}^{0}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{q}\mathfrak{n}}}(K,\mathbf{T}^*)^\vee\big)$$

Rearringing the sum,

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{n}\in\mathcal{N}_i} \mathrm{Fitt}^{0}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{n}}}(K,\mathbf{T}^*)^\vee\big)$$

By Theorem 6.2.10, we obtain that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) \lessapprox \sum_{\mathfrak{n}\in\mathcal{N}_i} \mathrm{ind}(\varepsilon_\mathfrak{n}) = \mathbf{\Theta}(\varepsilon) \qquad \square$$

6.3 Kolyvagin systems

In this section, we particularise the theory of Kolyvagin systems in §5.2 to compute Selmer groups with coefficients in the Iwasawa algebra.

6.3.1 Definition of Kolyvagin systems

Unlike Stark systems, Kolyvagin systems can be only defined when the core rank is positive.

Assumption (Pos). The core rank $\chi(\mathcal{F}) \geq 1$.

For every $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$, there are exact sequences

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_\mathfrak{u}}(K,\mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathrm{f}}(K_\mathfrak{u},\mathbf{T})$$

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_\mathfrak{u}}(K,\mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathrm{tr}}(K_\mathfrak{u},\mathbf{T})$$

Recall that, in the previous section, we had defined an isomorphism

$$\psi_{\mathfrak{u}}: \ \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, \mathbf{T}) \to R$$

Together with the finite-singular map defined in Proposition 5.1.34, it induces another isomorphism

$$\phi_{\mathfrak{u}} = \psi_{\mathfrak{u}} \circ \phi^{\mathrm{fs}}_{\mathfrak{u}}: \ \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, \mathbf{T}) \cong R$$

Under these identifications, the above exact sequences and Corollary B.4.9 induce maps

$$\alpha_{\mathfrak{n},\mathfrak{u}}: \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_{\mathfrak{u}}}(K, \mathbf{T})$$

$$\beta_{\mathfrak{n},\mathfrak{u}}: \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_{\mathfrak{u}}}(K, \mathbf{T})$$

Definition 6.3.1. A *Kolyvagin system* is a collection

$$(\kappa_{\mathfrak{n}})_{\mathfrak{n}\in\mathcal{N}} \in \prod_{\mathfrak{n}\in\mathcal{N}} \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \right)$$

satisfying, for all $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ not dividing $\mathfrak{n}$, that

$$\alpha_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}}) = \beta_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}})$$

Remark 6.3.2. When $\chi(\mathcal{F}) = 1$, Definition 5.2.1 can be used to construct a Kolyvagin system of Definition 6.3.1. Indeed, there is a natural map

$$\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}) \to \bigcap^{1} \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}) = \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T})^{++}$$

If we have a Kolyvagin system $\kappa \in \mathbf{KS}(\mathcal{F})$ as in Definition 5.2.1, we can consider the images $\kappa^{++}_{\mathfrak{n}}$ of $\kappa_{\mathfrak{n}}$ under the above map. One could see that

$$\alpha_{\mathfrak{n},\mathfrak{u}}(\kappa^{++}_{\mathfrak{n}}) = \phi_{\mathfrak{u}}(\mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}})), \quad \beta_{\mathfrak{n},\mathfrak{u}}(\kappa^{++}_{\mathfrak{n}\mathfrak{u}}) = \psi_{\mathfrak{u}}(\mathrm{loc}_{\mathfrak{u}}(\kappa_{\mathfrak{n}\mathfrak{u}}))$$

Only because $\phi_{\mathfrak{u}}$ and $\psi_{\mathfrak{u}}$ are defined satisfying $\phi_{\mathfrak{u}} = \psi_{\mathfrak{u}} \circ \phi^{\mathrm{fs}}_{\mathfrak{u}}$, we can see that Kolyvagin systems relations in Definitions 5.2.1 and 6.3.1 are equivalent.

6.3.2 Kolyvagin systems from Stark systems

In this section, we define a regulator map from the module of Stark systems to the Kolyvagin systems and prove that it is, indeed, an isomorphism.

For every $\mathfrak{n} \in \mathcal{N}$, there is an exact sequence

$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathcal{F}^n}(K, \mathbf{T}) \longrightarrow \prod_{\mathfrak{u}|n} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, \mathbf{T})$$

With the ordering of the primes previously defined, there is a well defined an isomorphism

$$\bigoplus_{\mathfrak{u}|n} \phi_{\mathfrak{u}} : \prod_{\mathfrak{u}|n} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, \mathbf{T}) \cong \Lambda^{\nu(n)}$$

Whith these identifications, Corollary B.4.9 constructs a map

$$(-1)^{\nu(\mathfrak{n})}\mathrm{Reg}_{\mathfrak{n}} : \bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$$

Theorem 6.3.3. Combined for all $n \in \mathcal{N}$, the maps Reg_n induce a regulator map

$$\mathrm{Reg} : \mathbf{SS}(\mathcal{F}) \to \mathbf{KS}(\mathcal{F})$$

Proof. We need to show that the image of the regulator of a Stark system satisfies the Kolyvagin system relations, i.e., that for every $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ not dividing $\mathfrak{n}$, we have that

$$\big(\alpha_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}} \circ \Phi_{\mathfrak{n}\mathfrak{u},\mathfrak{n}}\big)(\varepsilon_{\mathfrak{n}\mathfrak{u}}) = \big(\beta_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}\mathfrak{u}}\big)(\varepsilon_{\mathfrak{n}\mathfrak{u}})$$

since the Stark system relation implies that $\Phi_{\mathfrak{n}\mathfrak{u},\mathfrak{n}}(\varepsilon_{\mathfrak{n}\mathfrak{u}}) = \varepsilon_{\mathfrak{n}}$.

The maps $\big(\alpha_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}} \circ \Phi_{\mathfrak{n}\mathfrak{u},\mathfrak{n}}\big)$ and $\big(\beta_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}\mathfrak{u}}\big)$ can be computed using Corollary B.4.9 and the exact sequences

$$\mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K, \mathbf{T}) \rightarrowtail \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{u}}}(K, \mathbf{T}) \longrightarrow \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, \mathbf{T}) \oplus \bigoplus_{\substack{\mathfrak{q}|\mathfrak{n} \\ \mathfrak{q}=\mathfrak{u}}} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}}, \mathbf{T})$$

$$\mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K, \mathbf{T}) \rightarrowtail \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}\mathfrak{u}}}(K, \mathbf{T}) \longrightarrow \bigoplus_{\mathfrak{q}|\mathfrak{n}\mathfrak{u}} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}}, \mathbf{T}) \oplus \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}}, \mathbf{T})$$

It is important to remark what the subindex $\mathfrak{q} \mid \mathfrak{n}, \mathfrak{q} = \mathfrak{u}$ means: first we consider the summands where $\mathfrak{q} \mid \mathfrak{n}$, with the established order, and then we place the summand corresponding to the prime $\mathfrak{u}$. Note that the ordering matters to establish an isomorphism to $\Lambda^{\nu(\mathfrak{nu})+1}$, used in Corollary B.4.9 to construct the rank reduction maps[1]. With these ordering choices, the maps constructed by Corollary B.4.9 differ from $\left(\alpha_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}} \circ \Phi_{\mathfrak{nu},\mathfrak{n}}\right)$ and $\left(\beta_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{nu}}\right)$ in the signs of $(-1)^{\nu(\mathfrak{nu})}\mathrm{sign}(\mathfrak{nu},\mathfrak{n})$ and $(-1)^{\nu(\mathfrak{n})}$, respectively. This sign difference is due to a reordering of the summands, so we can conclude that

$$\left(\alpha_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{n}} \circ \Phi_{\mathfrak{nu},\mathfrak{n}}\right) = \left(\beta_{\mathfrak{n},\mathfrak{u}} \circ \mathrm{Reg}_{\mathfrak{nu}}\right)$$

□

It is especially important to note that the regulator maps are isomorphisms on core vertices.

Proposition 6.3.4. Let $\mathfrak{n} \in \mathcal{N}$ be a core vertex. Then the regulator map

$$\mathrm{Reg}_{\mathfrak{n}} : \bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$$

is an isomorphism.

Proof. By Proposition B.4.14,

$$\bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \cong \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \cong \Lambda$$

and the regulator of a generator of $\bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T})$ is an indivisible element in $\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$, so $\mathrm{Reg}_{\mathfrak{n}}$ is an isomorphism. □

6.3.3 The module of Kolyvagin systems

We will see in this section that the module of Kolyvagin systems is controlled by core vertices, similarly to the control made by weak core vertices to Stark systems in Theorem 6.2.6.

Theorem 6.3.5. Let $\mathfrak{n} \in \mathcal{N}$ be a core vertex. Then the projection map

$$\mathbf{KS}(\mathcal{F}) \to \bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$$

is an isomorphism.

[1]A different order of the summands might result in a sign difference when constructing the rank reduction maps.

The proof of Theorem 6.3.5 is more complicated than its analogousness in Theorem 6.2.6. We present it as a sequence of lemmas. We start by showing the surjectivity.

Lemma 6.3.6. Let $\mathfrak{n} \in \mathcal{N}$ be a core vertex. Then the projection map

$$\mathbf{KS}(\mathcal{F}) \to \bigcap^{\chi(\mathcal{F})+\nu(n)} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$$

is surjective.

Proof. Choose a core vertex $\mathfrak{n} \in \mathcal{N}$ and consider the following commutative diagram

$$\begin{tikzcd} \mathbf{SS}(\mathcal{F}) \arrow[r, "\mathrm{Reg}"] \arrow[d] & \mathbf{KS}(\mathcal{F}) \arrow[d] \\ \bigcap^{r+\nu(n)} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, \mathbf{T}) \arrow[r, "\mathrm{Reg}_{\mathfrak{n}}"] & \bigcap^{r} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \end{tikzcd}$$

By Theorem 6.2.6 and Proposition 6.3.4, the left vertical map and the bottom horizontal map are isomorphisms. Hence the right vertical map is surjective. $\square$

We next prove the injectivity of the map in Theorem 6.3.5. The argument takes a Kolyvagin system in the kernel of this map and proves inductively that all the classes κ_n have to vanish. We first prove the vanishing of κ_n, when n is another core vertex.

Lemma 6.3.7. Let $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ be elements such that $\mathfrak{n}$ and $\mathfrak{n}\mathfrak{u}$ are core vertices. Then there is an isomorphism

$$\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{u}} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K, T)$$

such that every $\kappa \in \mathrm{KS}(\mathcal{F})$ satisfies that $\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{u}}(\kappa_n) = \kappa_{\mathfrak{n}\mathfrak{u}}$.

Proof. Assume first that the corank of $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K, \mathbf{T}^*)$ is zero or, equivalently, that

$$\mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\big) \not\approx 0$$

Since $\mathfrak{n}$ is a core vertex, Proposition 4.3.11 gives an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_{\mathfrak{u}}}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T}) \arrow[r, "\mathrm{loc}_{\mathfrak{u}}"] & \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T}) \arrow[dll, rounded corners] \\
& \mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)^\vee \arrow[r] & 0
\end{tikzcd}
$$

Then the image of $\mathrm{loc}_{\mathfrak{u}}\big(\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})\big)$ has Λ-rank one, so Propositions 6.1.3 and B.4.14 imply that

$$
\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T}) \cong \Lambda \cong \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})
$$

Therefore, the image of $\alpha_{\mathfrak{n},\mathfrak{u}}$ is the unique principal submodule of $\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})$ such that

$$
\mathrm{Im}(\alpha_{\mathfrak{n},\mathfrak{u}}) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big) \cdot \left(\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})\right)
$$

Similarly, since $\mathfrak{n}\mathfrak{u}$ is a core vertex, there is another exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})_{\mathfrak{u}}}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathrm{tr}}(K_{\mathfrak{u}},\mathbf{T}) \arrow[dll, rounded corners] \\
& \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})^{\mathfrak{u}}}(K,\mathbf{T}^*)^\vee \arrow[r] & 0
\end{tikzcd}
$$

The same argument leads to show that $\beta_{\mathfrak{n},\mathfrak{u}}$ is injective and its the unique submodule of $\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})$ such that

$$
\mathrm{Im}(\beta_{\mathfrak{n},\mathfrak{u}}) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big) \left(\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})\right)
$$

Therefore, $\mathrm{Im}(\alpha_{\mathfrak{n},\mathfrak{u}}) = \mathrm{Im}(\beta_{\mathfrak{n},\mathfrak{u}})$ and $\beta_{\mathfrak{n},\mathfrak{u}}$ is injective, so the following map is well defined and an isomorphism:

$$
\gamma_{\mathfrak{n},\mathfrak{u}} := \beta_{\mathfrak{n},\mathfrak{u}}^{-1} \circ \alpha_{\mathfrak{n},\mathfrak{u}}
$$

The definition of Kolyvagin system implies that $\gamma_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}}) = \kappa_{\mathfrak{n}\mathfrak{u}}$ for all Kolyvagin systems $\kappa \in \mathbf{KS}(\mathcal{F})$.

Assume now that the corank of $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)$ is one. By Proposition 5.1.44, we can find an ultraprime $\mathfrak{q}\in\mathcal{P}$ such that the corank of $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}_{\mathfrak{q}}(\mathfrak{n})}(K,\mathbf{T}^*)$ is zero and the map

$$\mathrm{loc}_{\mathfrak{q}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})\to\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}},\mathbf{T})$$

is surjective. The surjectivity of $\mathrm{loc}_{\mathfrak{q}}$ implies that $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{q}}(\mathfrak{n})}(K,\mathbf{T}^*)$ vanishes, so we can define the map $\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{q}}$ as above.

On the other hand, the corank of $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)$ being one and Proposition 4.3.11 imply that the following maps are zero:

$$\mathrm{loc}_{\mathfrak{u}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})\to\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T})$$
$$\mathrm{loc}_{\mathfrak{u}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T})\to\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T})$$

Hence $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})=\mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})=\mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T})$. Then the map

$$\mathrm{loc}_{\mathfrak{q}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T})\to\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}},\mathbf{T})$$

is also surjective, so $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{q}}(\mathfrak{n}\mathfrak{u})}(K,\mathbf{T}^*)=0$ and the map $\gamma_{\mathfrak{n}\mathfrak{u},\mathfrak{n}\mathfrak{u}\mathfrak{q}}$ is well defined.

Finally, the map

$$\mathrm{loc}_{\mathfrak{q}}:\ \mathbf{H}^1_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})\to\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}},\mathbf{T})$$

is surjective too. Thus Lemma 5.1.38 implies that

$$\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n}\mathfrak{q})}(K,\mathbf{T}^*)=\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}_{\mathfrak{q}}(\mathfrak{n})}(K,\mathbf{T}^*)$$

By the assumption on the ultraprime $\mathfrak{q}\in\mathcal{P}$, we have that the Selmer group $\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n}\mathfrak{q})}(K,\mathbf{T}^*)$ has corank zero, so $\gamma_{\mathfrak{n}\mathfrak{q},\mathfrak{n}\mathfrak{u}\mathfrak{q}}$ is well defined. We then define the map

$$\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{u}}=\gamma^{-1}_{\mathfrak{n}\mathfrak{u},\mathfrak{n}\mathfrak{u}\mathfrak{q}}\circ\gamma_{\mathfrak{n}\mathfrak{q},\mathfrak{n}\mathfrak{u}\mathfrak{q}}\circ\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{q}}:\ \bigcap^{\chi(\mathcal{F})}\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})\to\bigcap^{\chi(\mathcal{F})}\mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K,T)$$

By construction, every Kolyvagin system $\kappa\in\mathbf{KS}(\mathcal{F})$ satisfies that $\gamma_{\mathfrak{n},\mathfrak{u}}(\kappa_{\mathfrak{n}})=\kappa_{\mathfrak{n}\mathfrak{u}}$. □

Remark 6.3.8. One can show that the map $\gamma_{\mathfrak{n},\mathfrak{u}}$ is independent of the choice of the auxiliary prime $\mathfrak{q}$. However, we don't make use of this fact and only need the existence of such map for our purposes.

We can improve Lemma 6.3.7 to extend the definition of the maps γ to other vertices. In this extension, those maps are no longer isomorphisms between the whole exterior biduals but a submodule of them.

Definition 6.3.9. Let $\mathcal{N}_0 \subset \mathcal{N}$ be the subset of vertices defined as

$$\mathcal{N}_0 := \left\{\mathfrak{n} \in \mathcal{N} : \ \mathrm{rank}_\Lambda \ \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)^\vee = 0\right\}$$

Note that, by Propositions 5.1.37, 5.1.4 and B.4.14, we have that

$$\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \cong \Lambda \ \forall \mathfrak{n} \in \mathcal{N}_0$$

For every $\mathfrak{n} \in \mathcal{N}_0$, we define the module $\Upsilon(\mathfrak{n})$ as the unique cyclic submodule of $\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$ such that

$$\Upsilon(\mathfrak{n}) \approx \left(\mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\right)\right) \cdot \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})\right)$$

In an abuse of notation, we will sometimes say that $\Upsilon(\mathfrak{n}) = 0$ whenever $\mathfrak{n} \notin \mathcal{N}_0$.

Lemma 6.3.10. Assume that we have vertices $\mathfrak{n}, \mathfrak{n}\mathfrak{u} \in \mathcal{N}_0$. Then we can define an isomorphisms

$$\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{u}} : \ \Upsilon(\mathfrak{n}) \to \Upsilon(\mathfrak{n}\mathfrak{u})$$

satisfying that, for every $\kappa \in \mathrm{KS}(\mathcal{F})$, if $\kappa_\mathfrak{n} \in \Upsilon(\mathfrak{n})$, then $\gamma_{\mathfrak{n},\mathfrak{n}\mathfrak{u}}(\kappa_\mathfrak{n}) = \kappa_{\mathfrak{n}\mathfrak{u}}$.

Proof. Assume first that the corank of $\mathbf{H}^1_{(\mathcal{F}^*)^\mathfrak{u}(\mathfrak{n})}(K, \mathbf{T}^*)$ is zero.

Since $\bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_\mathfrak{u}(\mathfrak{n})}(K, \mathbf{T}) \cong \Lambda$ by Propositions 5.1.37, 5.1.4 and B.4.14, we can define the submodule $\Omega_{\mathfrak{n},\mathfrak{u}}$ as the unique cyclic submodule satisfying that

$$\Omega_{\mathfrak{n},\mathfrak{u}} \approx \left(\mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{\mathcal{F}^\mathfrak{u}(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\right)\right) \cdot \left(\bigcap^{\chi_\mathcal{F}-1} \mathbf{H}^1_{\mathcal{F}_\mathfrak{u}(\mathfrak{n})}(K, \mathbf{T})\right)$$

Since there is a surjective map

$$\mathbf{H}^1_{(\mathcal{F}^*)^\mathfrak{u}(\mathfrak{n})}(k, \mathbf{T}^*)^\vee \twoheadrightarrow \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(k, \mathbf{T}^*)^\vee$$

then, for every height one prime ideal β, there is an inclusion

$$\mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{(\mathcal{F}^*)^\mathfrak{u}(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\right)_\beta \subset \mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\right)_\beta$$

In addition, if $C_{\mathfrak{n},\mathfrak{u}}$ denotes the cokernel of the map

$$\mathrm{loc}_\mathfrak{u} : \ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \mathbf{H}^1_\mathrm{f}(K_\mathfrak{u}, T)$$

then Proposition 4.3.11 and Corollary B.2.10 imply that

$$\mathrm{Fitt}^0_\Lambda\big(C_{\mathfrak{n},\mathfrak{u}}\big)\mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big)$$

Assume first that the corank of $\mathbf{H}^1_{\mathcal{F}^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})$ is zero. By Proposition B.4.14, the map $\alpha_{\mathfrak{n},\mathfrak{u}}$ restricted to the following subspaces is an isomorphism

$$\alpha_{\mathfrak{n},\mathfrak{u}}:\ \Upsilon(\mathfrak{n}) \to \Omega_{\mathfrak{n},\mathfrak{u}}$$

Similarly, we have another isomorphism

$$\beta_{\mathfrak{n},\mathfrak{u}}:\ \Upsilon(\mathfrak{nu}) \to \Omega_{\mathfrak{n},\mathfrak{u}}$$

We can thus define the isomorphism

$$\gamma_{\mathfrak{n},\mathfrak{nu}} := \beta^{-1}_{\mathfrak{n},\mathfrak{u}} \circ \alpha_{\mathfrak{n},\mathfrak{u}}:\ \Upsilon(\mathfrak{n}) \to \Upsilon(\mathfrak{nu})$$

When the corank of $\mathbf{H}^1_{\mathcal{F}^{\mathfrak{u}}(\mathfrak{n})}(K,\mathbf{T})$ is one, construct $\mathfrak{q} \in \mathcal{P}$ as in Lemma 6.3.7. The same argument leads to

$$\mathrm{rank}_\Lambda\ \mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{q}}(\mathfrak{n})}(K,\mathbf{T}^*)^\vee = \mathrm{rank}_\Lambda\ \mathbf{H}^1_{(\mathcal{F}^*)^{\mathfrak{u}}(\mathfrak{nq})}(K,\mathbf{T}^*)^\vee = 0$$

We can then construct

$$\gamma_{\mathfrak{n},\mathfrak{nu}} = \gamma^{-1}_{\mathfrak{nu},\mathfrak{nuq}} \circ \gamma_{\mathfrak{nq},\mathfrak{nuq}} \circ \gamma_{\mathfrak{n},\mathfrak{nq}}:\ \Upsilon(\mathfrak{n}) \to \Upsilon(\mathfrak{nu})$$

By construction, we have that $\gamma_{\mathfrak{n},\mathfrak{nu}}(\kappa_{\mathfrak{n}}) = \kappa_{\mathfrak{nu}}$ for every $\kappa \in \mathbf{KS}(\mathcal{F})$. $\square$

Lemma 6.3.11. Assume $p \geq 5$, let $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathcal{P}$ and let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}_0$ be vertices such that $\mathfrak{q}_1 \mid \mathfrak{n}_1$ and $\mathfrak{q}_2 \mid \mathfrak{n}_2$. Then there is an ultraprime $\mathfrak{u} \in \mathcal{P}$ such that both $\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i$ are core vertices for $i = 1, 2$ and there are isomorphisms

$$\gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i}:\ \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i)}(K,\mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i)}(K,\mathbf{T})$$

such that, for all $\kappa \in \mathbf{KS}(\mathcal{F})$, the equality $\gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i}(\kappa_{\mathfrak{n}}) = \kappa_{\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i}$.

Proof. We can apply Proposition 5.1.39 to find $\mathfrak{u} \in \mathcal{P}$ such that the maps

$$\mathrm{loc}_{\mathfrak{u}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T})\ \forall i = 1,2$$

are surjective for $i = 1, 2$. We can further assume that, whenever $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)_{\mathfrak{q}_i}}(K,T)$ has positive rank, the map

$$\mathrm{loc}_{\mathfrak{u}}:\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i/\mathfrak{q}_i)_{\mathfrak{q}_i}}(K,\mathbf{T}) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T})$$

is surjective. In addition, we can impose that the localisation maps

$$\mathrm{loc}_{\mathfrak{u}}^{*}:\ \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T}^*)$$

are injective. Note that we can choose such ultraprime $\mathfrak{u} \in \mathcal{P}$ since $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*)[\mathfrak{m}]$ is a one dimensional k-vector space whenever $\mathfrak{n}_i/\mathfrak{q}_i$ is not a core vertex. Indeed, it can be seen from the exact sequence

$$\mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{q}_i},\mathbf{T}^*)^\vee \longrightarrow \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*)^\vee \twoheadrightarrow \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)_{\mathfrak{q}_i}}(K,\mathbf{T}^*)^\vee$$

Since $\mathfrak{q}_i$ is a Kolyvagin ultraprime, the first term in the previous exact sequence is isomorphic to Λ, and the last term vanishes because $\mathfrak{n}_i$ is a core vertex. Therefore, $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*)^\vee$ is a quotient of Λ and hence $\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*)[\mathfrak{m}]$ is at most one dimensional as k-vector space.

The surjectivity of $\mathrm{loc}_{\mathfrak{u}}$ together with the injectivity of $\mathrm{loc}_{\mathfrak{u}}^*$ implies that $\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i$ is a core vertex, since by Lemma 5.1.38 we have that

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i)}(K,\mathbf{T}^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}}(\mathfrak{n}_i/\mathfrak{q}_i)}(K,\mathbf{T}^*) = 0$$

When $\mathrm{rank}_\Lambda\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i/\mathfrak{q}_i)_{\mathfrak{q}_i}}(K,T) \geq 1$, the choice of $\mathfrak{u}$ and Lemma 5.1.38 imply that $\mathfrak{n}_i\mathfrak{u}$ is a core vertex for $i = 1,2$ since

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i\mathfrak{u})}(K,\mathbf{T}^*) = \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i)_{\mathfrak{u}}}(K,\mathbf{T}^*) = 0$$

In that case, we can construct the map $\gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i}$ as the composition of maps constructed in Lemma 6.3.11, i.e.,

$$\gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i} = \gamma^{-1}_{\mathfrak{n}_i/\mathfrak{q}_i\mathfrak{u},\mathfrak{n}_i\mathfrak{u}} \circ \gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}}$$

When $\mathrm{rank}_\Lambda\ \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_i/\mathfrak{q}_i)_{\mathfrak{q}_i}}(K,\mathbf{T}) = 0$, then we necessarily have, by Proposition 5.1.37, that $\chi(\mathcal{F}) = 1$ and that

$$\mathrm{rank}_\Lambda\ \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)^{\mathfrak{q}_i}}(K,T^*)^\vee = \mathrm{rank}_\Lambda\ \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n}_i/\mathfrak{q}_i)}(K,T^*)^\vee = 0$$

We can now use the maps in Lemma 6.3.10, to construct

$$\gamma_{\mathfrak{n}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i} = \gamma_{\mathfrak{n}_i/\mathfrak{q}_i,\mathfrak{n}_i\mathfrak{u}/\mathfrak{q}_i} \circ \gamma^{-1}_{\mathfrak{n}_i/\mathfrak{q}_i,\mathfrak{n}_i} \qquad \square$$

We can now use the previous two lemmas to construct the map to construct such maps for any two core vertices.

Lemma 6.3.12. Let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}_0$. Then there is an isomorphism

$$\gamma_{\mathfrak{n}_1,\mathfrak{n}_2} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_1)}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_2)}(K, \mathbf{T})$$

such that, for every $\kappa \in \mathbf{KS}(\mathcal{F})$, then $\gamma_{\mathfrak{n}_1,\mathfrak{n}_2}(\kappa_{\mathfrak{n}_1}) = \kappa_{\mathfrak{n}_2}$.

Proof. We can assume without loss of generality that the quantity $s = \nu(\mathfrak{n}_2) - \nu(\mathfrak{n}_1) \geq 0$. A recursive application of Corollary 5.1.42 shows the existence of primes $\mathfrak{u}_1, \ldots, \mathfrak{u}_s$ such that for all i, the product $\mathfrak{n}\mathfrak{u}_1 \cdots \mathfrak{u}_i \in \mathcal{N}_0$. Let $\mathfrak{r} := \mathfrak{n}\mathfrak{u}_1 \cdots \mathfrak{u}_s$. By Lemma 6.3.10, we can construct by composition a map

$$\gamma_{\mathfrak{n}_1,\mathfrak{r}} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_1)}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{r})}(K, \mathbf{T})$$

For every prime $\mathfrak{q}_1 \mid \mathfrak{r}$ not dividing $\mathfrak{n}_2$ and every prime $\mathfrak{q}_2 \mid \mathfrak{n}_2$ and not dividing $\mathfrak{r}$, we can apply Lemma 6.3.11 recursively to find the map

$$\gamma_{\mathfrak{r},\mathfrak{n}_2} : \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{r})}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}_2)}(K, \mathbf{T})$$

The isomorphism $\gamma_{\mathfrak{n}_1,\mathfrak{n}_2}$ is constructed as the composition of these maps. $\square$

Corollary 6.3.13. Let $\kappa \in \mathbf{KS}(\mathcal{F})$ be a Kolyvagin system. If $\kappa_{\mathfrak{c}} = 0$ for some core vertex $\mathfrak{c} \in \mathcal{N}$, then $\kappa_{\mathfrak{n}} = 0$ for any other vertices $\mathfrak{n} \in \mathcal{N}_0$. In particular, $\kappa_{\mathfrak{n}} = 0$ for all other core vertices $\mathfrak{n}$.

Lemma 6.3.14. Let $\kappa \in \mathrm{KS}(\mathcal{F})$ be a Kolyvagin system such that $\kappa_{\mathfrak{c}} = 0$ for some core vertex $\mathfrak{c} \in \mathcal{N}$. Then $\kappa = 0$.

Proof. We will show that $\kappa_{\mathfrak{n}} = 0$ for all $\mathfrak{n} \in \mathcal{N}$. We will proceed by induction on the quantity

$$\lambda(\mathfrak{n}) = \mathrm{rank}_\Lambda\, \mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee$$

When $\lambda(\mathfrak{n}) = 0$, then we can deduce by Lemma 6.3.12 that $\kappa_{\mathfrak{n}} = 0$.

Assume now that $\kappa_{\mathfrak{r}} = 0$ for all $\mathfrak{r} \in \mathcal{N}$ such that $\lambda(\mathfrak{r}) \leq k$ for some natural number k and that $\mathfrak{n} \in \mathcal{N}$ is a vertex satisfying that $\lambda(\mathfrak{n}) = k$ and that $\kappa_{\mathfrak{n}} \neq 0$.

By Propositions 5.1.44 and B.4.16, we can construct a Kolyvagin ultraprime such that

$$\kappa_{\mathfrak{n}} \notin \ker \left(\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \bigcap^{\chi(\mathcal{F})-1} \mathbf{H}^1_{\mathcal{F}_\tau(\mathfrak{n})}(K, \mathbf{T}) \right)$$

and $\lambda(\mathfrak{n}\tau) < \lambda(\mathfrak{n})$. By the induction hypothesis, $\kappa_{\mathfrak{n}\tau} = 0$, which contradicts the Kolyvagin relation between $\kappa_{\mathfrak{n}}$ and $\kappa_{\mathfrak{n}\tau}$. The contradiction proves that $\kappa_{\mathfrak{n}} = 0$ for all $\kappa_n \in \mathcal{N}$. $\square$

Proof of Theorem 6.3.5. By Lemma 6.3.6 the projection

$$\mathbf{KS}(\mathcal{F}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$$

is surjective. In addition, Lemma 6.3.14 proves the injectivity. $\square$

Hence we know that the module of Kolyvagin systems is a free, cyclic Λ-module that is controlled by the module of Stark systems.

Corollary 6.3.15. Let $\mathcal{F}$ be a Selmer structure of core rank $\chi(\mathcal{F}) \geq 1$. Then the set of Kolyvagin systems is a free, cyclic Λ-module.

Corollary 6.3.16. The regulator map

$$\mathrm{Reg}: \ \mathbf{SS}(\mathcal{F}) \to \mathbf{KS}(\mathcal{F})$$

is an isomorphism.

It now makes sense to consider generators of the module of Kolyvagin systems.

Definition 6.3.17. We say that a Kolyvagin system $\kappa \in \mathrm{KS}(\mathcal{F})$ is *primitive* if it generates $\mathbf{KS}(\mathcal{F})$ as a Λ-module.

Analogously to Theorem 6.2.10, primitive Kolyvagin systems control the Fitting ideal with transverse local conditions.

Theorem 6.3.18. Suppose Assumption (BI) holds and that $\mathcal{F}$ is a Selmer structure of core rank $\chi(\mathcal{F}) \geq 1$. If $\kappa \in \mathbf{KS}(\mathcal{F})$ is a primitive Kolyvagin system, then

$$\mathrm{ind}(\kappa_{\mathfrak{n}}) \gtrapprox \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)^\vee\big) \ \forall\, \mathfrak{n} \in \mathcal{N}$$

Proof. By Corollary 6.3.16, there is a primitive Stark system ε such that $\mathrm{Reg}(\varepsilon) = \kappa$. For every $\mathfrak{n} \in \mathcal{N}$, Proposition 6.1.6 shows the existence of a weak core vertex $\mathfrak{c} \in \mathcal{N}$ such that $\mathfrak{n} \mid \mathfrak{c}$. Then Proposition 2.2.6 gives an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T}) \arrow[dll, rounded corners] \\
\bigoplus\limits_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}},\mathbf{T}) \oplus \bigoplus\limits_{\mathfrak{u}|\mathfrak{c}/\mathfrak{n}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T}) \arrow[r] & \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K,\mathbf{T}^*)^\vee \arrow[r] & 0
\end{tikzcd}
$$

Then $\kappa_{\mathfrak{n}} = (\mathrm{Reg}_{\mathfrak{n}} \circ \phi_{\mathfrak{c},\mathfrak{n}})(\varepsilon_{\mathfrak{c}})$. Since ε is a primitive Stark system, Theorem 6.2.6 shows that $\varepsilon_{\mathfrak{c}}$ generates $\bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{c})} \mathbf{H}^1_{\mathcal{F}^c}(K,\mathbf{T})$. Hence Propositions B.4.14 and B.4.15 imply that

$$\mathrm{ind}(\kappa_n) \gtrapprox \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big) \qquad \square$$

6.3.4 Fitting ideals of the Selmer group

In this section, we give a description of the Fitting ideals of the Selmer group, by relating them with the theta ideals of a primitive Kolyvagin systems.

Theorem 6.3.19. Let $\kappa \in \mathbf{KS}(\mathcal{F})$ be a primitive Kolyvagin system. Then

$$\mathbf{\Theta}(\kappa) \gtrapprox \mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big)$$

Proof. By Corollary 5.1.42, we can choose a weak core vertex $\mathfrak{c} \in \mathcal{N}$ such that, for every $\mathfrak{u} \mid \mathfrak{c}$, the following equality holds

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{u})}(K,\mathbf{T}^*) = \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{u}}}(K,\mathbf{T}^*) \tag{6.5}$$

The core vertex $\mathfrak{c}$ induces an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T}) \arrow[r] & \bigoplus\limits_{\mathfrak{u}|\mathfrak{c}} \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}},\mathbf{T}) \arrow[dll, rounded corners] \\
& \mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee \arrow[r] & 0
\end{tikzcd}
$$

For every $\mathfrak{q} \mid \mathfrak{c}$, there is another exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{q}}}(K,\mathbf{T}) \arrow[r] & \mathbf{H}^1_{\mathcal{F}^{\mathfrak{c}}}(K,\mathbf{T}) \arrow[r] & \bigoplus\limits_{\mathfrak{u}|\mathfrak{n}/\mathfrak{q}} \mathbf{H}^1_{s}(K_{\mathfrak{u}},\mathbf{T}) \arrow[dll, rounded corners] \\
& \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{q}}}(k,\mathbf{T}^*) \arrow[r] & 0
\end{tikzcd}
$$

By the definition of the Fitting ideals, we can deduce that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}|\mathfrak{c}} \mathrm{Fitt}^{i-1}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)_\mathfrak{q}}(K,\mathbf{T}^*)^\vee\big)$$

By (6.5), we can deduce that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}|\mathfrak{c}} \mathrm{Fitt}^{i-1}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{q})}(K,\mathbf{T}^*)^\vee\big)$$

By Proposition 6.1.10, we can extend the equality to all the ultraprimes in $\mathcal{P}$:

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}\in\mathcal{P}} \mathrm{Fitt}^{i-1}_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{q})}(K,\mathbf{T}^*)^\vee\big)$$

We can now proceed to prove Theorem 6.3.19 by induction. For $i=0$, it is Theorem 6.3.18 for $n=1$. Assuming the result holds true for $i-1$, for all Selmer structures $\mathcal{F}(\mathfrak{q})$, we have that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{q}\in\mathcal{P}} \sum_{\substack{\mathfrak{n}\in\mathcal{N}_{i-1}\\ \mathfrak{q}\nmid\mathfrak{n}}} \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{q}\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big)$$

Rearringing the sum,

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) = \sum_{\mathfrak{n}\in\mathcal{N}_i} \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K,\mathbf{T}^*)^\vee\big)$$

By Theorem 6.3.18, we obtain that

$$\mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big) \approx \sum_{\mathfrak{n}\in\mathcal{N}_i} \mathrm{ind}(\kappa_\mathfrak{n}) = \mathbf{\Theta}(\kappa) \qquad \square$$

Then the propagation of $\mathcal{F}$ to $\mathbf{T}/(X)$ is also cartesian, the theta ideals coincide exactly with the theta ideals.

Corollary 6.3.20. Let $\mathcal{F}$ be a cartesian Selmer structure on $\mathbf{T}$ whose projection to $\mathbf{T}/(X)$ is also cartesian. If $\kappa \in \mathbf{KS}(\mathbf{T},\mathcal{F})$ is a primitive Kolyvagin system, and Assumptions (BI) and (H2) hold, then

$$\mathbf{\Theta}(\kappa) = \mathrm{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K,\mathbf{T}^*)^\vee\big)$$

Proof. Note that Assumptions (Free) and (FC) hold in this case due to Proposition 6.1.15 and Corollary 6.1.18. Then the equality is obtained from Corollary 5.2.31 and Theorem 6.3.19 $\square$

Knowing this equivalence for ultra Kolyvagin systems can be used to deduce its analogousness for the classical Kolyvagin systems with coefficients in self-injective quotients of Λ.

Theorem 6.3.21. Let $\mathcal{F}$ be a cartesian Selmer structure on $\mathbf{T}$ whose projection to $\mathbf{T}/(X)$ is also cartesian and let R be a self-injective quotient of the Iwasawa algebra such that the propagation of $\mathcal{F}$ to $T := \mathbf{T} \otimes_\Lambda R$ is cartesian. If $\kappa' \in \mathbf{KS}(\mathbf{T} \otimes_\Lambda R, \mathcal{F})$ is a primitive Kolyvagin system, and Assumptions (BI) and (H2) hold, then

$$\mathbf{\Theta}(\kappa') = \mathrm{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Proof. Note that Proposition 6.1.15 and Corollary 6.1.18 imply Assumptions (Free) and (FC).

Since κ' is primitive, its projection to $\mathrm{KS}(T/\mathfrak{m}, \mathcal{F})$ is non-zero, so Theorem 5.2.19 guarantees the existence of a primitive Kolyvagin system $\kappa \in \mathbf{KS}(T, \mathcal{F})$ projecting to κ', which is understood as an ultra Kolyvagin systems via the identification in Theorem 5.2.8.

By Corollaries 5.2.24 and 5.2.27, we have that

$$\mathbf{\Theta}(\kappa) + I \subset \mathbf{\Theta}(\kappa')$$

where I is the kernel of the map $\Lambda \twoheadrightarrow R$. Similarly, Proposition 5.1.17 implies that

$$\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee = \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \otimes_\Lambda R$$

Therefore,

$$\mathrm{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big) = \mathrm{Fitt}^i_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee\big) + I$$

Using Theorem 6.3.19, we can conclude that

$$\mathrm{Fitt}^i_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T)^\vee\big) \subset \mathbf{\Theta}(\kappa')$$

The equality then follows using [BSS25, Theorem 5.2]. $\square$

Remark 6.3.22. The main impediment to show Theorem 6.3.21 was the difficulty to prove that the Selmer group $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$ contains a free R-submodule for certain values $\mathfrak{n} \in \mathcal{N}$. It seems plausible that the formalism of ultraprimes can be used to show the existence of these submodules for infinitely many $\mathfrak{n}$, leading to a different prove of Theorem 6.3.21 using the theory in [BSS25].

6.4 Structure of the Selmer group

Assume that $\mathcal{F}$ is a cartesian Selmer structure. Since the Selmer group $\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T})$ is finitely generated, the structure theorem of Iwasawa modules shows there is a pseudo-isomorphism

$$\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee \approx \Lambda^r \times \prod_{i=1}^{s} \Lambda/(p)^{\alpha_i} \times \prod_{j=1}^{t} \Lambda/(f_j)$$

where r, s, t, α_i are non-negative integers such that $\alpha_1 \geq \cdots \geq \alpha_s$ and the f_i are distinguished polynomials such that $f_s \mid \cdots \mid f_1$. By convention, we will set $\alpha_i = 0$ for $i \geq s$ and $f_j = 1$ for all $j \geq t$.

Let $\kappa \in \mathbf{KS}(\mathcal{F})$ and $\varepsilon \in \mathbf{SS}(\mathcal{F})$ be primitive Kolyvagin and Stark systems, respectively. By Theorems 6.2.11 and 6.3.19, we can define $\mathbf{\Theta}$ as the unique principal ideal of Λ such that

$$\mathbf{\Theta} \approx \mathbf{\Theta}(\kappa) \approx \mathbf{\Theta}(\varepsilon) \approx \mathrm{Fitt}^i_{\Lambda}\big(\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}^*)^\vee\big)$$

By Corollary B.2.9, we know that

$$\begin{aligned} \mathbf{\Theta} &= 0 && \text{if } i < r \\ \mathbf{\Theta} &= \prod_{k > r-i} (p^{\alpha_k} f_k) && \text{if } i \geq r \end{aligned}$$

We can use this information to recover the invariants associated with the Selmer group.

Corollary 6.4.1. The rank r of $\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}^*)^\vee$ is the minimal $i \in \mathbb{Z}_{\geq 0}$ such that $\mathbf{\Theta} \neq 0$.

Corollary 6.4.2. For every $i \in \mathbb{Z}_{\geq 0}$, we have that $\Theta_{i+1} \mid \mathbf{\Theta}$. In addition, if $i \geq r$, we have that

$$\frac{\Theta_{i+1}}{\mathbf{\Theta}} = p^{\alpha_{i-r}} f_{i-r}$$

where the quotient is taken as fractional ideals. Since (f_{i-r}) is not divisible by (p), the quotient determines both α_{i-r} and f_{i-r}.

Therefore, the knowledge of the ideals $\mathbf{\Theta}$ determines $\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}^*)^\vee$ up to pseudo-isomorphism of Iwasawa modules. In addition, Fitting ideals can see more information about the structure of the Selmer groups, like its finite submodules. The following is a generalisation of [Kur14a, Proposition 2.9], which assumed the vanishing of the Iwasawa μ-invariant associated with the Selmer group.

Theorem 6.4.3. Let $\mathcal{F}$ be a cartesian Selmer structure on $\mathbf{T}$ of positive core rank whose propagation to $\mathbf{T}/(X)$ is also cartesian. If $\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee$ is a torsion Iwasawa module, then it contains no finite submodules.

Proof. By Proposition B.2.5, we need to prove that

$$\mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)\big)^\vee$$

is a principal ideal. By Corollary 6.3.20, this Fitting ideal coincides with $\mathrm{ind}(\delta_1)$. By Proposition 6.1.3, then $\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T})$ is a free module of rank $\chi(\mathcal{F})$. Since

$$\delta_1 \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}) \cong \Lambda$$

then $\mathrm{ind}(\delta_1)$ is principal. □

6.5 A control theorem

Assume that $\mathbf{T}$ is obtained as a tensor product

$$\mathbf{T} = T \otimes_{\mathbb{Z}_p} \Lambda$$

for some free, finitely generated $\mathbb{Z}_p$-module of finite rank, endowed with an action of the Galois group G_K. Then the quotient with the Λ-ideal generated by X induces an isomorphism

$$\mathbf{T} \otimes_\Lambda \Lambda/(X) = T$$

This is the principal process to construct Galois representations over the Iwasawa algebra, motivated by Shapiro's lemma. Then the importance of study how does Kolyvagin systems behave under the projection map $\mathbf{T} \twoheadrightarrow \mathbf{T}/(X)$.

Proposition 6.5.1. Let $\mathcal{F}$ be a cartesian Selmer structure on $\mathbf{T}$ such that its propagation to $\mathbf{T}/(X)$ is also cartesian. Then there is a canonical map

$$\Pi : \ \mathbf{KS}(\mathbf{T}, \mathcal{F}) \to \mathbf{KS}(\mathbf{T}/(X), \mathcal{F})$$

Proof. Let $\mathfrak{n} \in \mathcal{N}$. Since $\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$ is a free module by Proposition 6.1.15, so there is a homomorphism

$$\bigcap_{\Lambda}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \to \bigcap_{\Lambda/(X)}^{\chi(\mathcal{F})} \frac{\mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})}{(X)} \to \bigcap_{\Lambda/(X)}^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}\left(K, \frac{\mathbf{T}}{(X)}\right)$$

By Proposition B.4.10, the product of these maps for all $\mathfrak{n} \in \mathcal{N}$ restricts to a homomorphism

$$\Pi : \ \mathbf{KS}(\mathbf{T}, \mathcal{F}) \to \mathbf{KS}(\mathbf{T}/(X), \mathcal{F})$$

□

It is important to compute the cokernel of the map Π in order to study the primality of Kolyvagin systems. Using the Kolyvagin derivative process described in Chapter A, we obtain Kolyvagin systems for both $\mathbf{T}$ and T, related under the projection Π. The primality of the Kolyvagin system for $\mathbf{T}$ is equivalent to the Iwasawa main conjecture, which is known in some cases of interest. Thus, the primality of the Kolyvagin system for T is equivalent to the surjectivity of Π.

When $\mathcal{F}$ is the canonical Selmer structure and $\chi(\mathcal{F})$, the following is a result of K. Büyükboduk in [Büy11, Theorem A], considering the identifications in 5.2.19.

Theorem 6.5.2. Under Assumption (H2), the projection map

$$\Pi : \ \mathbf{KS}(\mathbf{T}, \mathcal{F}) \to \mathbf{KS}(\mathbf{T}/(X), \mathcal{F})$$

is surjective.

Proof. By 6.1.10, we can find some core vertex $\mathfrak{n} \in \mathcal{N}(\mathbf{T})$. By the definition of Kolyvagin primes, we know that $\mathfrak{n} \in \mathcal{N}(T)$. By Proposition 5.1.17, we can compute

$$\mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, (\mathbf{T}/(X))^*) \cong \mathbf{H}^1_{(\mathcal{F}^*)(\mathfrak{n})}(K, \mathbf{T}^*)[X] = 0$$

Hence $\mathfrak{n}$ is a core vertex for $\mathbf{H}^1_{\mathcal{F}}(K, \mathbf{T}/(X))$. By Theorem 6.3.5, we can consider the following commutative diagram with vertical isomorphisms:

$$\begin{tikzcd}
\mathrm{KS}(\mathbf{T}, \mathcal{F}) \arrow[r, "\Pi"] \arrow[d, "\sim"] & \mathrm{KS}(\mathbf{T}/(X), \mathcal{F}) \arrow[d, "\sim"] \\
\bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}) \arrow[r] & \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T}/(X))
\end{tikzcd}$$

We can mimic the argument in Proposition 6.1.15 to compute the cokernel C of the bottom map. Since $\mathcal{F}$ is cartesian, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\frac{\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T})}{(X)} \arrow[r] \arrow[d] & \frac{\mathbf{H}^1(K^{\Sigma_{\mathcal{F}(\mathfrak{n})}}/K,\mathbf{T})}{(X)} \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}(\mathfrak{n})}} \frac{\mathbf{H}^1_{/\mathcal{F}(\mathfrak{n})}(K,\mathbf{T})}{(X)} \arrow[d] \\
\mathbf{H}^1_{\mathcal{F}}\left(K,\frac{\mathbf{T}}{(X)}\right) \arrow[r] & \mathbf{H}^1\left(K^{\Sigma_{\mathcal{F}(\mathfrak{n})}}/K,\frac{\mathbf{T}}{(X)}\right) \arrow[r] & \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}(\mathfrak{n})}} \mathbf{H}^1_{/\mathcal{F}(\mathfrak{n})}\left(K,\frac{\mathbf{T}}{(X)}\right)
\end{tikzcd}$$

The snake lemma induces a short exact sequence

$$0 \longrightarrow C \longrightarrow \mathbf{H}^2(K^{\mathfrak{n}}/K,\mathbf{T})[X] \longrightarrow \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}(\mathfrak{n})}} \mathbf{H}^2(K,\mathbf{T})[X] \longrightarrow 0$$

By Proposition 4.2.29, there is another short exact sequence

$$\mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{n}}}(K,\mathbf{T}^*)^\vee \rightarrowtail \mathbf{H}^2(K^{\mathfrak{n}}/K,\mathbf{T}) \twoheadrightarrow \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}(\mathfrak{n})}} \mathbf{H}^2(K,\mathbf{T})$$

Comparing both short exact sequences, there is an isomorphism

$$C \cong \mathbf{H}^1_{(\mathcal{F}^*)_{\mathfrak{n}}}(K,\mathbf{T}^*)^\vee[X] = 0$$

where we are using the fact that $\mathfrak{n}$ is a core vertex. Therefore, the projection map Π is surjective. $\square$

Corollary 6.5.3. Let $\kappa \in \mathbf{KS}(\mathbf{T},\mathcal{F})$ be a primitive Kolyvagin system. Then $\Pi(\kappa)$ is also a primitive Kolyvagin system in $\mathbf{KS}(T,\mathcal{F})$.

Remark 6.5.4. When T is the Tate module of an elliptic curve whose Tamagawa product is divisible by p, there is a conjecture of C.H. Kim (see [Kim25, Conjecture 1.9])stating that the cokernel of the map in Theorem 6.5.2 is the p-adic valuation of the Tamagawa product. Note that, in this case, Assumption (H2) does not hold. This conjecture has been proven under mild assumptions in [Bur+26] and [CS26].

The theory in this section will be used to prove the following theorem, at the end of Chapter A below.

Theorem 6.5.5. Assume that T is a Galois representation satisfying Assumption (BI) such that $T \otimes_{\mathbb{Z}_p} \Lambda$ satisfies Assumption (H2). In addition, let $c \in \mathrm{ES}(T)$ be an Euler system satisfying the Iwasawa main conjecture, as in Definition A.2.3Then the Kolyvagin system $\kappa \in \mathrm{KS}(T,\mathcal{F})$ obtained from the Euler system in Corollary A.4.11 below is primitive.

Remark 6.5.6. It is expected that the map Π is not surjective whenever Assumption (H2) does not hold, but its cokernel has length coinciding with the p-adic valuation of the Tamagawa numbers. That has been proven in [Bur+26] for the Tate module of an elliptic curve.

Part III

Cartesian systems

Chapter 7

Cartesian systems

In this chapter, we generalise the ideas behind the constructions of Kolyvagin and Stark systems in order to develop a theory of cartesian systems.

Both Kolyvagin and Stark systems in Chapter 6 are collections of classes in the exterior biduals of different cartesian Selmer structures of a specific form. The Kolyvagin system and Stark system relation could be described as compatibility under the maps constructed in Proposition B.4.8.

In this chapter, we find out that the same argument can be used to consider the collection of classes living in every cartesian Selmer group that can be defined on the Galois representation. It leads to the notion of cartesian systems.

Analogously to the theory of Kolyvagin and Stark systems, cartesian systems can be controlled by core Selmer structures, so they are a free module of rank one. In addition, the different elements in a primitive cartesian system can measure the 0^{th} Fitting ideal of each Selmer group.

Stark and Kolyvagin systems can be obtained from cartesian systems by restricting them to the appropriate set of Selmer structures. They are examples of partial cartesian systems.

In §7.3, we explore under which conditions cartesian systems can be extended to a (total) cartesian system, so their elements establish upper bounds on the Fitting ideals of the Selmer groups.

It is expected that this theory will be the starting point of a Kolyvagin system formalism under less restrictive hypotheses.

7.1 Cartesian Selmer structures

This section gives a characterization of cartesian Selmer structures over principal rings, both artinian rings and discrete valuation rings, and over the Iwasawa algebra Λ. The definition of cartesian structure is more complicated when R is artinian (see Definition 2.2.11) than when R is a discrete valuation ring or the Iwasawa algebra (see Definition 5.1.5). However, we will see that both definitions are essentially equivalent.

Definition 7.1.1. Let $\mathcal{F} \leq \mathcal{G}$ be two Selmer structures. Its *local quotient* $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is defined as

$$\mathcal{L}_{\mathcal{G}/\mathcal{F}} := \bigoplus_{\mathfrak{u} \in \mathcal{U}(\mathbb{P})} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) = \bigoplus_{\mathfrak{u} \in \mathcal{U}(\mathbb{P})} \frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)}$$

We will show that the cartesian assumption on $\mathcal{F}$ and $\mathcal{G}$ implies that the local quotient is a free module whenever the coefficient ring is principal, and pseudo-isomorphic to a free module when the coefficient ring is the Iwasawa algebra. In addition, the rank of the local quotient is the difference of the core ranks $\chi(\mathcal{G}) - \chi(\mathcal{F})$.

7.1.1 Coefficients in self-injective rings

This section computes the local quotient of two cartesian Selmer structures when the coefficient ring R is self-injective. We are using the patched cohomology setting for homogeneous reasons with the rest of the chapter, although it makes no difference when R is finite.

The first result of this section proves the freeness of the local quotients between two cartesian Selmer structures. It can be seen as a generalisation of Proposition 3.3.3.

Proposition 7.1.2. Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures. Then the local quotient

$$\mathcal{L}_{\mathcal{G}/\mathcal{F}} = \bigoplus_{\mathfrak{u} \in \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)$$

is a free R-module of rank $\chi(\mathcal{G}) - \chi(\mathcal{F})$.

Proof. Let $\mathfrak{n}$ be the square-free product of all the ultraprimes in $\Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}}$. By Proposition 2.2.6, there is an global-duality exact sequence for $T \otimes k$:

$$0 \to \mathbf{H}^1_{\mathcal{F}}(K, T\otimes k) \longrightarrow \mathbf{H}^1_{\mathcal{G}}(K, T\otimes k) \to \bigoplus_{\mathfrak{u}|\mathfrak{n}} \frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{q}}, T\otimes k)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{q}}, T\otimes k)} \to \mathbf{H}^1_{\mathcal{F}^*}(K, T^*[\mathfrak{m}])^\vee \to \mathbf{H}^1_{\mathcal{G}^*}(K, T^*[\mathfrak{m}])^\vee \longrightarrow 0$$

Definition 2.2.14 and dimension counting imply that

$$\dim_k \left(\bigoplus_{\mathfrak{u}|\mathfrak{n}} \frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{q}}, T\otimes k)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{q}}, T\otimes k)} \right) = \chi(\mathcal{G}) - \chi(\mathcal{F})$$

Lemma 7.1.4 below implies that

$$\bigoplus_{\mathfrak{u}|\mathfrak{n}} \Big(\ell_R\big(\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big) - \ell_R\big(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big)\Big) = \ell_R(\mathfrak{m}^i)(\chi(\mathcal{G}) - \chi(\mathcal{F})) \tag{7.1}$$

In particular, for $i = 0$, we obtain that

$$\# \left(\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \right) = \big(\#R\big) \cdot (\chi(\mathcal{G}) - \chi(\mathcal{F}))$$

Consider the following composition, which coincides with the multiplication by a generator x of the ideal $R[\mathfrak{m}]$.

$$\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \longrightarrow \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T\otimes k) \xrightarrow{x} \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)$$

The first map is surjective by the propagation of Selmer structures and the second one is injective since $\mathcal{F}$ is cartesian. Hence there is a short exact sequence

$$0 \longrightarrow \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T)[R[\mathfrak{m}]] \longrightarrow \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \to \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T\otimes k) \longrightarrow 0$$

We can then deduce that

$$\# \left(\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\ell}, T)[R[\mathfrak{m}]] \right) = \frac{\# \left(\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\ell}, T) \right)}{\# \left(\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\ell}, T\otimes k) \right)}$$

In conclusion, $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ has length equal to

$$\operatorname{length}(R)(\chi(\mathcal{G}) - \chi(\mathcal{F}))$$

and whose $R[\mathfrak{m}]$-torsion induces a non-trivial quotient of length $\chi(\mathcal{G}) - \chi(\mathcal{F})$. Hence, the structure theorem implies that

$$\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \cong R^{\chi(\mathcal{G})-\chi(\mathcal{F})} \qquad \square$$

We now state two technical lemmas used in the previous proof.

Lemma 7.1.3. Let $\mathcal{F}$ be a cartesian Selmer structure. Then there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{H}^0(K_{\mathfrak{u}}, \mathfrak{m}^{i+1}T) \arrow[r] & \mathbf{H}^0(K_{\mathfrak{u}}, \mathfrak{m}^{i}T) \arrow[r] & \mathbf{H}^0(K_{\mathfrak{u}}, T \otimes k)^{\omega_i} \arrow[dll, out=0, in=180] & \\ & \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^{i+1}T) \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^{i}T) \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k)^{\omega_i} \arrow[r] & 0 \end{tikzcd}$$

where $\omega_i = \operatorname{rank}_R(T)(\ell_R(\mathfrak{m}^i) - \ell_R(\mathfrak{m}^{i+1}))$.

Proof. It follows from the exact sequence in Proposition 4.2.6 (see also Remark 4.2.8) applied to the short exact sequence:

$$0 \longrightarrow \mathfrak{m}^{i+1}T \longrightarrow \mathfrak{m}^{i}T \longrightarrow (T \otimes k)^{\omega_i} \longrightarrow 0 \tag{7.2}$$

which is exact because T is a free R-module.

It is important to remark that the image of the connecting homomorphism is contained in $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^{i+1}T)$ by Definition 2.2.8, which forces all the elements in the kernel of the bottom, left, horizontal map to belong to the local condition. The exactness at $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^{i}T)$ follows from Definition 2.2.8 too.

The exactness of the right requires using the cartesian condition. Note that, in the construction of the exact sequence, we have implicitly used an isomorphism

$$(T \otimes k)^{\omega_i} \to \frac{\mathfrak{m}^i T}{\mathfrak{m}^{i+1}T} \Rightarrow \mathbf{H}^i(K_{\mathfrak{u}}, T \otimes k)^{\omega_k} \to \mathbf{H}^i\left(K_{\mathfrak{u}}, \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}\right)$$

This isomorphism has an explicit description. Let $\{e_1, \ldots, e_{\omega_i}\}$ be a minimal set of generators of $\mathfrak{m}_i$. Note that Nakayama's lemma implies that $\{e_1 + \mathfrak{m}^{i+1}, \ldots, e_{\omega_i} + \mathfrak{m}^{i+1}\}$ forms a basis of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$

as a k-vector space. Then the isomorphism can be obtained by multiplying the j^{th} component by e_j.

We will show that

$$c \in \mathbf{H}^1_{\mathcal{F}}\left(K_{\mathfrak{u}}, \frac{T}{\mathfrak{m}}\right) \Leftrightarrow$$

The implication $\Rightarrow$ follows from Definition 2.2.7, the fact that local conditions are R-modules and the diagram:

$$\begin{tikzcd}
\mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[r, "e_k"] \arrow[d] & \mathbf{H}^1(K_{\mathfrak{u}}, T) \arrow[d] \\
\mathbf{H}^1(K_{\mathfrak{u}}, T \otimes k) \arrow[r, "{[e_k]}"] & \mathbf{H}^1(K_{\mathfrak{u}}, T \otimes R/\mathfrak{m}^{i+1})
\end{tikzcd}$$

The implication $\Leftarrow$ is equivalent to the injectivity of the map

$$[e_j] : \ \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}) \to \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^{i+1})$$

It can be deduced since $\mathcal{F}$ satisfies Definition 2.2.11. Let x be a generator of $R[\mathfrak{m}] = \mathfrak{m}^{a-1}$, where a is defined as the minimum natural number such that $\mathfrak{m}^a = 0$. Since $R[\mathfrak{m}] \subset e_jR$, there exists some $y \in R$ such that $ye_j = x$. Since $e_j \notin m^{i+1}$, we can deduce that $y \in \mathfrak{m}^{a-1-y}$. Then, the injective map $[x]$ from Definition 2.2.11 coincides with the composition $[x] = [y] \circ [e_j]$, where $[y]$ is the map

$$[y] : \ \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T/\mathfrak{m}^{i+1}) \to \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T)$$

which is well-defined because $y \in \mathfrak{m}^{a-1-y}$. Since $[x]$ is injective, then $[e_j]$ is also injective.

In addition, Definition 2.2.7 implies that

$$e_jc \in \mathbf{H}^1_{\mathcal{F}}\left(K_{\mathfrak{u}}, \frac{T}{\mathfrak{m}^{i+1}}\right) \Leftrightarrow e_jc \in \mathbf{H}^1_{\mathcal{F}}\left(K_{\mathfrak{u}}, \frac{\mathfrak{m}^i T}{\mathfrak{m}^{i+1}}\right)$$

Hence, the last term can be identified with

$$\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k)^{\omega_i} = \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \frac{\mathfrak{m}_i T}{\mathfrak{m}^{i+1} T})$$

and the exactness on the right follows from Definition 2.2.7. $\square$

Lemma 7.1.4. Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures and let $\mathfrak{u}$ be an ultraprime. Then there is some $\alpha \in \mathbb{Z}_{\geq 0}$ such that

$$\ell_R\big(\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big) - \ell_R\big(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big) = \alpha \ell_R(\mathfrak{m}^i)$$

where ℓ_R denotes the length as R-modules.

Proof. We will proceed by reversed induction, since the lemma holds for all $i \in \mathbb{N}$ such that $\mathfrak{m}^i = 0$. Let

$$\alpha := \ell_R\big(\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T \otimes k)\big) - \ell_R\big(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k)\big)$$

In addition, denote

$$d_i := \ell_R\big(\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big) - \ell_R\big(\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, \mathfrak{m}^i T)\big)$$

Assume the result holds for some $i \in \mathbb{N}$, i.e., that $d_i = \alpha \ell_R(\mathfrak{m}^i)$. Since $\mathcal{F}$ and $\mathcal{G}$ are cartesian, the exact sequences from Lemma 7.1.3 and the additivity of the length implies that

$$d_{i-1} = d_i + \alpha \omega_{i-1}$$

By the induction hypothesis,

$$d_{i-1} = \alpha\big(\ell_R(\mathfrak{m}^i) + \omega_{i-1}\big)$$

From the exact sequence in (7.2), we can deduce that

$$d_{i-1} = \alpha \ell_R(\mathfrak{m}^{i-1})$$

□

The converse of Proposition 7.1.2 also holds true: if the local quotient is a free module, then $\mathcal{F}$ is cartesian if and only if $\mathcal{G}$ is cartesian. Hence the next results show that we can obtain cartesian Selmer structures by 'adding' or 'removing' free copies of R.

Proposition 7.1.5. Let R be a self-injective, local ring and let $\mathcal{G}$ be a cartesian Selmer structure. Consider another Selmer structure $\mathcal{F} \leq \mathcal{G}$ such that there is an ultraprime $\mathfrak{u}$ such that

$$\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{q}}, T) = \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{q}}, T) \ \forall \mathfrak{q} \neq \mathfrak{u}$$

Assume further that

$$\mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \cong R$$

Then, $\mathcal{F}$ is also cartesian.

Proof. Since all other local conditions coincide, we only need to check the cartesian property at the local condition at $\mathfrak{u}$. Since R is a projective R-module, there is a decomposition

$$\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) \cong \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \oplus R$$

We will see that the image of the free summand under the projection

$$\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) \to \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T \otimes k)$$

is a one-dimensional k-vector space. Indeed, the dimension of the image is bounded above by one because R is generated by one element. It is also non-zero because the kernel is contained in $H^1(K_\ell, T)[\mathfrak{m}^{k-1}]$, since the following composition is the multiplication by π^{k-1}, where π is a generator of $\mathfrak{m}$:

$$\bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T) \longrightarrow \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T \otimes k) \xrightarrow{\pi^{k-1}} \bigoplus_{\mathfrak{u}|\mathfrak{n}} \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$$

Therefore,

$$\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T \otimes k) = \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k) \oplus k$$

Since R is self-injective, the structure theorem gives decompositions,

$$\begin{aligned} \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T) &= \mathbf{H}^1_{/\mathcal{G}}(K_{\mathfrak{u}}, T) \oplus R \\ \mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k) &= \mathbf{H}^1_{/\mathcal{G}}(K_{\mathfrak{u}}, T \otimes k) \oplus k \end{aligned}$$

The map

$$\mathbf{H}^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T \otimes k) \to H^1_{/\mathcal{F}}(K_{\mathfrak{u}}, T)$$

respects the above decomposition, so it is injective because $\mathcal{G}$ is cartesian. Hence $\mathcal{F}$ is also cartesian. □

Proposition 7.1.6. Let R be a self-injective, local ring and let $\mathcal{F}$ be a cartesian Selmer structure. Consider another Selmer structure $\mathcal{G} \geq \mathcal{F}$ such that there is a prime $\mathfrak{u}$ such that

$$\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{q}}, T) = \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{q}}, T) \ \forall \mathfrak{q} \neq \mathfrak{u}$$

Assume further that

$$\mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_{\mathfrak{u}}, T) \cong R$$

Then, $\mathcal{G}$ is also cartesian.

Proof. Analogously to the proof of Proposition 7.1.5, there are decompositions

$$\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, T) = \mathbf{H}^1_{/\mathcal{G}}(K_\mathfrak{u}, T) \oplus R$$
$$\mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, T \otimes k) = \mathbf{H}^1_{/\mathcal{G}}(K_\mathfrak{u}, T \otimes k) \oplus k$$

Since the map in Definition 2.2.11 respects this decomposition, the cartesian property of $\mathcal{F}$ implies that the map

$$\mathbf{H}^1_{/\mathcal{G}}(K_\mathfrak{u}, T \otimes k) \hookrightarrow \mathbf{H}^1_{/\mathcal{G}}(K_\mathfrak{u}, T)$$

is injective, so $\mathcal{G}$ is also cartesian. $\square$

7.1.2 Coefficients in discrete valuation ring

We can show an analogue of Proposition 7.1.2 when the coefficient ring R is a discrete valuation ring. Since the definition of cartesian structures in Definition 5.1.5 is simpler, it is also easier to study the local quotients in this setting. The following result is a generalisation of Proposition 5.4.1.

Proposition 7.1.7. Assume that the coefficient ring R is a discrete valuation ring. If $\mathcal{F}$ and $\mathcal{G}$ are cartesian Selmer structures, the local quotient is a free module of rank $\chi(\mathcal{G}) - \chi(\mathcal{F})$.

Proof. Since $\mathcal{F}$ is cartesian, the R-module

$$\bigoplus_{\mathfrak{u} \in \mathcal{U}(\mathbb{P})} \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, T)$$

is torsion-free. Since $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is a finitely-generated submodule, it is free by the structure theorem. By Proposition 4.3.11, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K, T) \arrow[r] & \mathbf{H}^1_{\mathcal{G}}(K, T) \arrow[r] & \mathcal{L}_{\mathcal{G}/\mathcal{F}} \arrow[dll, out=0, in=180] \\
& \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \arrow[r] & \mathbf{H}^1_{\mathcal{G}^*}(K, T^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

By Definition 5.1.18 and the additivity of the rank, the rank of $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is $\chi(\mathcal{G}) - \chi(\mathcal{F})$. $\square$

Remark 7.1.8. The analogue of Propositions 7.1.5 and 7.1.6 does not hold in this case. The free, rank one modules needs to be chosen maximal, so the induced quotients do not contain any torsion elements.

7.1.3 Coefficients in the Iwasawa algebra

In Selmer structures defined for Galois representations over the Iwasawa algebra, the local quotients might no longer be free modules. However, it is always a torsion-free finitely generated Iwasawa module, so pseudo-isomorphic to a free Λ-module.

Proposition 7.1.9. Let $\mathcal{F}$ and $\mathcal{G}$ are cartesian Selmer structures defined on a Galois representation whose coefficient ring is the Iwasawa algebra. Then the local quotient $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is pseudo-isomorphic to a free Λ-module. Counting the ranks in the global duality exact sequence from Proposition 4.3.11, the rank of $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ coincides with $\chi(\mathcal{G}) - \chi(\mathcal{F})$.

Proof. The local quotient $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is a submodule of

$$\bigoplus_{\mathfrak{u}\in\mathcal{U}(\mathbb{P})} \mathbf{H}^1_{/\mathcal{F}}(K_\mathfrak{u}, T)$$

which is Λ-torsion free. By the structure theorem of finitely generated Iwasawa modules, the local quotient $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is pseudo-isomorphic to $\Lambda^{\chi(\mathcal{G})-\chi(\mathcal{F})}$. $\square$

7.2 Cartesian systems

In this section, we use the study of the local quotients done in the previous section to construct cartesian systems. This section assumes that the coefficient ring R is an integral domain satisfying Assumption (R-lim).

7.2.1 Local quotients

When $\mathcal{F} \leq \mathcal{G}$ are cartesian Selmer structures, recall that we have defined in Definition 7.2.4 the local quotient as

$$\mathcal{L}_{\mathcal{G}/\mathcal{F}} := \bigoplus_{\mathfrak{u}\in\mathcal{U}(\mathbb{P})} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K_\mathfrak{u}, T) = \bigoplus_{\mathfrak{u}\in\mathcal{U}(\mathbb{P})} \frac{\mathbf{H}^1_{\mathcal{G}}(K_\mathfrak{u}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_\mathfrak{u}, T)}$$

which is a free, finitely generated R-module of rank $\chi(\mathcal{G}) - \chi(\mathcal{F})$. We want to extend the definition of $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ for every pair of Selmer structures $\mathcal{F}$ and $\mathcal{G}$, not necessarily comparable. In order to do that, we need to consider the intersection of Selmer structures.

It is technically important that the intersection of two cartesian Selmer structures is also cartesian.

Definition 7.2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be Selmer structures. The *intersection* $\mathcal{F} \cap \mathcal{G}$ is the Selmer structure defined by the local conditions

$$\mathbf{H}^1_{\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}}, T) := \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$$

Proposition 7.2.2. Assume R is an integral domain satisfying Assumption (R-lim). If $\mathcal{F}$ and $\mathcal{G}$ are cartesian Selmer structures, the intersection $\mathcal{F} \cap \mathcal{G}$ is also cartesian.

Proof. Let $\mathfrak{u}$ be an ultraprime. We need to see that the local condition $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$ is cartesian, i.e., that the quotient

$$\frac{\mathbf{H}^1(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)}$$

is torsion-free.

Let $c \in \mathbf{H}^1(K_{\mathfrak{u}}, T)$ and let $\alpha \in R \setminus \{0\}$ satisfying that $\alpha c \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$ for some natural number n. Since both $\mathcal{F}$ and $\mathcal{G}$ are cartesian, Definition 5.1.5 implies that

$$c \in \mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$$

Hence, $\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T) \cap \mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)$, so the set theoretic intersection coincides with the cartesian intersection. $\square$

Remark 7.2.3. Proposition 7.2.2 is no longer true when the coefficient ring R is not an integral domain. The reason behind this is that the intersection of two free R-modules is not necessarily free. That is the reason why we only consider integral domains in this section.

We can now extend the definition of local quotient to any two cartesian Selmer structures, not necessarily related.

Definition 7.2.4. Let $\mathcal{F}$ and $\mathcal{G}$ be cartesian Selmer structures. We define the *local quotient* as

$$\mathcal{L}_{\mathcal{G}/\mathcal{F}} := \bigoplus_{\mathfrak{u}\in\Sigma_{\mathcal{F}}\cup\Sigma_{\mathcal{G}}} \left[\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}}, T)} \oplus \left(\frac{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}}, T)}{\mathbf{H}^1_{\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}}, T)} \right)^+ \right]$$

7.2.2 The graph of cartesian Selmer structures

In this section, we limit the scope of the theory, assuming that R is either a discrete valuation ring, or the Iwasawa algebra.

When $\mathcal{F} \leq \mathcal{G}$, the determinant of the local quotient fits into the following map between biduals of Selmer groups.

Proposition 7.2.5. Let $\mathcal{F} \leq \mathcal{G}$ be cartesian Selmer structures. There is a canonical homomorphism

$$\phi_{\mathcal{G},\mathcal{F}} : \bigcap^{\chi(\mathcal{G})} \mathbf{H}^1_{\mathcal{G}}(K,T) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)$$

Proof. If R is a discrete valuation ring, $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is a free R-module of rank $\chi(\mathcal{G}) - \chi(\mathcal{F})$, so Proposition B.4.7 constructs the map $\phi_{\mathcal{G},\mathcal{F}}$.

When R is the Iwasawa algebra, $\mathcal{L}_{\mathcal{G}/\mathcal{F}}$ is only pseudo-isomorphic to the Iwasawa algebra, but the map $\phi_{\mathcal{G},\mathcal{F}}$ can be constructed with Proposition B.4.8. $\square$

The existence of this map motivates the following definition.

Definition 7.2.6. We consider the directed graph $\mathfrak{C}$ whose vertices are the cartesian Selmer structures with non-negative core rank on T and there is an arrow $\mathcal{G} \to \mathcal{F}$ joining two Selmer structures $\mathcal{F}$ and $\mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$.

Definition 7.2.7. Fix a base cartesian Selmer structure $\mathcal{F}_0$. For every cartesian Selmer structure $\mathcal{F}$, define

$$\mathbf{X}(\mathcal{F}) = \mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) := \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$$

Proposition 7.2.8. Let $\mathcal{F} \leq \mathcal{G}$ be two cartesian Selmer structures. Then there is a map

$$\Phi_{\mathcal{G}/\mathcal{F}} : \mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

In order to prove this result, we need the following lemma.

Lemma 7.2.9. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}_0$ be cartesian Selmer structures such that $\mathcal{F} \leq \mathcal{G}$. Then there is a canonical isomorphism,

$$\det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}) = \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$$

Proof. Denote $\Sigma := \Sigma_{\mathcal{F}} \cup \Sigma_{\mathcal{G}} \cup \Sigma_{\mathcal{F}_0}$. We can split the determinant of $\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}$ as

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

Here, all the summands are free R-modules, so we can identify them with their biduals. Hence we can use Propositions B.4.3 and B.4.4 to split the determinants. Then the determinant $\det\big(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}\big)$ can be expressed as

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)\otimes$$
$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

We can use Proposition B.4.4 to simplify and express $\det\big(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}\big)$ as

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

Similarly, $\det\big(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}\big)$ can be expressed as

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

Same computation express $\det\big(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0}\big)$ as

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

By Proposition B.4.4, we have that

$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{F}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)=R$$

Therefore,

$$\det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}})\otimes\det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})=$$
$$\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\left(\frac{\mathbf{H}^1_{\mathcal{G}}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)^+\right)\otimes\det\left(\bigoplus_{\mathfrak{u}\in\Sigma}\frac{\mathbf{H}^1_{\mathcal{F}_0}(K_{\mathfrak{u}},T)}{\mathbf{H}^1_{\mathcal{F}_0\cap\mathcal{F}\cap\mathcal{G}}(K_{\mathfrak{u}},T)}\right)$$

Thus,

$$\det\big(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}\big)=\det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}})\otimes\det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$$ □

Remark 7.2.10. The choices on the sign conventions in Lemma 7.2.9 deserve some comment. We impose that, whenever we introduce or cancel a pair of a free summand and its dual to the direct sum, that is always done in consecutive spots with the dual group on the left position. Then the sign is independent of the position of the pair.

Under these sign conventions, for any cartesian Selmer structures $\mathcal{F} \leq \mathcal{G} \leq \mathcal{H}$, the following isomorphisms coincide:

$$\begin{tikzcd}
\det(\mathcal{L}^+_{\mathcal{H}/\mathcal{F}_0}) \arrow[r] & \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0}) \arrow[d] \\
& \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{G}}) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})
\end{tikzcd}$$

$$\begin{tikzcd}
\det(\mathcal{L}^+_{\mathcal{H}/\mathcal{F}_0}) \arrow[r] & \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{G}}) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}_0}) \arrow[d] \\
& \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{G}}) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})
\end{tikzcd}$$

Proof of Proposition 7.2.8. By Lemma 7.2.9, the tensor product of the map in Proposition 7.2.5 with the identity map on $\det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$ leads to the desired map $\Phi_{\mathcal{G}/\mathcal{F}}$. $\square$

Proposition 7.2.11. The assignment $\mathcal{F} \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ forms an inverse system indexed by the set of cartesian Selmer structures.

Proof. We need to show that, whenever $\mathcal{F} \leq \mathcal{G} \leq \mathcal{H}$ are cartesian Selmer structures, then

$$\Phi_{\mathcal{H},\mathcal{F}} = \Phi_{\mathcal{G},\mathcal{F}} \circ \Phi_{\mathcal{H},\mathcal{G}}$$

By Proposition B.4.11, we have $\phi_{\mathcal{H},\mathcal{F}} = \phi_{\mathcal{G},\mathcal{F}} \circ \phi_{\mathcal{H},\mathcal{G}}$ with the sign identification given by the map

$$\det(\mathcal{L}^+_{\mathcal{H}/\mathcal{G}}) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) = \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{F}}),\ e_{\mathcal{H}/\mathcal{G}} \otimes e_{\mathcal{G}/\mathcal{F}} \mapsto e_{\mathcal{G}/\mathcal{F}} \wedge e_{\mathcal{H}/\mathcal{G}}$$

After tensoring with $\det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$, we obtain the relation $\Phi_{\mathcal{H},\mathcal{F}} = \Phi_{\mathcal{G},\mathcal{F}} \circ \Phi_{\mathcal{H},\mathcal{G}}$ under the sign convention

$$\det(\mathcal{L}^+_{\mathcal{H}/\mathcal{G}}) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0}) = \det(\mathcal{L}^+_{\mathcal{H}/\mathcal{F}}) \otimes \det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$$

These sign conventions coincide with the ones used in Lemma 7.2.9 (see Remark 7.2.10) in order to constuct the maps Φ. Hence, the commutative relation holds for every triple of Selmer structures. $\square$

It means that $X_{\mathcal{F}_0}(\mathcal{F})$ forms an inverse system indexed by the cartesian Selmer structures defined on T. Hence it makes sense to consider the elements in the inverse limit.

Definition 7.2.12. We define the set of *(total) cartesian systems* as the elements in the inverse limit

$$\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) = \varprojlim_{\mathcal{F}\in\mathfrak{C}} \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

where the limit is taken over all cartesian Selmer structures.

Remark 7.2.13. In order to define cartesian systems, we needed to fix a base Selmer structure $\mathcal{F}_0$. However, the choice of $\mathcal{F}_0$ is not important for the theory. Indeed, if we have choose a different base $\mathcal{G}_0$, there is a canonical isomorphism

$$\mathrm{CART}_{\mathcal{G}_0}(\mathfrak{C}) \cong \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \otimes \det(\mathcal{L}^{+}_{\mathcal{F}_0/\mathcal{G}_0})$$

7.2.3 The module of cartesian systems

We saw in the theory of Stark and Kolyvagin system there were some special vertices, the core vertices, that controlled the Stark and Kolyvagin systems. A similar situation happens with cartesian systems, which are controlled by their values on some specific structures, the core Selmer structures.

Definition 7.2.14. A cartesian Selmer structure is called a *core structure* when

$$H^1_{\mathcal{F}^*}(K, T^*) = 0$$

Proposition 7.2.15. Under Assumptions (BI), for every cartesian Selmer structure $\mathcal{F}$, there is core structure $\mathcal{G}$ such that $\mathcal{F} \leq \mathcal{G}$.

Proof. It follows from the existence of weak core vertices, as shown in 6.1.6 $\square$

Lemma 7.2.16. Let $\mathcal{F}$ be a core Selmer structure. Then $\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ is a free R-module of rank one.

Proof. The proof of this proposition will be different depending on R being a discrete valuation ring or the Iwasawa algebra. Assume first

that R is a discrete valuation ring, Definition 5.1.18 and Proposition 5.1.4 prove that
$$\mathbf{H}^1_{\mathcal{F}}(K,T) \cong R^{\chi(\mathcal{F})}$$
Then,
$$\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \cong \bigcap^{\chi(\mathcal{F})} H^1_{\mathcal{F}}(K,T) \cong R$$
When R is the Iwasawa algebra, we can no longer describe $\mathbf{H}^1_{\mathcal{F}}(K,T)$ up to isomorphism, but up to pseudo-isomorphism. Indeed, Definition 5.1.18 and Proposition 6.1.3 show that
$$\mathbf{H}^1_{\mathcal{F}}(K,\mathbf{T}) \approx \Lambda^{\chi(\mathcal{F})}$$
By Lemma B.3.2, we obtain that
$$\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \cong \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T) \cong \bigwedge^{\chi(\mathcal{F})} \Lambda^{\chi(\mathcal{F})} = \Lambda \qquad \square$$

Proposition 7.2.17. Let $\mathcal{F} \leq \mathcal{G}$ be two core Selmer structures. Then the map
$$\Phi_{\mathcal{G},\mathcal{F}}: \ \mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$
is an isomorphism.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are core Selmer structures, Proposition 4.3.11 induces an exact sequence
$$0 \longrightarrow \mathbf{H}^1_{\mathcal{F}}(K,T) \longrightarrow \mathbf{H}^1_{\mathcal{G}}(K,T) \longrightarrow \mathcal{L}_{\mathcal{G}/\mathcal{F}} \longrightarrow 0$$
By Lemma 7.2.16,
$$\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \cong \mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \cong R$$
Then Propositions B.4.13 and B.4.14 imply that
$$\phi_{\mathcal{G}/\mathcal{F}}: \bigcap^{\chi(\mathcal{G})} \mathbf{H}^1_{\mathcal{G}}(K,T) \otimes \det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)$$
is surjective, so it has to be an isomorphism.

By Lemma 7.2.9, the map $\Phi_{\mathcal{G},\mathcal{F}}$ is the tensor product of $\phi_{\mathcal{G},\mathcal{F}}$ with the identity map in $\det(\mathcal{L}^+_{\mathcal{F}/\mathcal{F}_0})$, so it is also an isomorphisms. $\square$

Similarly to what happened with Kolyvagin and Stark systems, cartesian systems can be controlled by core Selmer structures.

Proposition 7.2.18. Let $\mathcal{F}$ be a core Selmer structure. Then the map

$$\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

is an isomorphism.

Proof. Let $c \in \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$. We will show that there is a unique $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ such that $\xi_{\mathcal{F}} = c$.

Let $\mathcal{G}$ be a cartesian structure. By Proposition 7.2.15, there is a core Selmer structure $\mathcal{H}$ such that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{G} \leq \mathcal{H}$. Since the map $\Phi_{\mathcal{H},\mathcal{F}} : \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}) \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ is an isomorphism by Proposition 7.2.17, the definition of the inverse limit implies that $\xi_{\mathcal{G}}$ is the image of the $\xi_{\mathcal{F}}$ under the map

$$\Phi_{\mathcal{H},\mathcal{G}} \circ \Phi_{\mathcal{H},\mathcal{F}}^{-1} : \ \mathbf{X}(\mathcal{F}) \to \mathbf{X}(\mathcal{G}) : \ \xi_{\mathcal{F}} \to \xi_{\mathcal{G}}$$

Hence, the element is determined by c. Moreover, this construction defines an element of $\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$, so the projection map is an isomorphism. $\square$

Corollary 7.2.19. The module of cartesian systems $\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ is a free, cyclic R-module.

It now makes sense to consider the generators of this module.

Definition 7.2.20. A cartesian system $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ is called *primitive* if it generates $\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ as an R-module.

We can now apply Propositions B.4.13 and B.4.14 to relate the elements of cartesian systems with the Fitting ideals of the dual Selmer group.

Theorem 7.2.21. Let $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ be a primitive cartesian system and let $\mathcal{F}$ be a cartesian Selmer structure. If R is a discrete valuation ring, then

$$\mathrm{ind}(\xi_{\mathcal{F}}) = \mathrm{Fitt}_R^0\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Otherwise, when R is the Iwasawa algebra, the above is only true up to pseudo-equality:

$$\mathrm{ind}(\xi_{\mathcal{F}}) \approx \mathrm{Fitt}_\Lambda^0\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee\big)$$

Proof. Let $\mathcal{G} \geq \mathcal{F}$ be a core Selmer structure. Since ξ is primitive, Proposition 7.2.18 implies that $\xi_{\mathcal{G}}$ generates $\mathbf{X}_{\mathcal{F}_0}(\mathcal{G})$. Since $\xi_{\mathcal{F}} = \Phi_{\mathcal{G},\mathcal{F}}(\xi_{\mathcal{G}})$, the result follows from either Proposition B.4.13 or Proposition B.4.14, depending on whether R is a discrete valuation ring or the Iwasawa algebra. $\square$

7.3 Partial cartesian systems

In practice, constructing a cartesian system is complicated, and we are only able to construct classes for a subgraph of the cartesian Selmer structures. The goal of this section will be to establish conditions on the subgraph that guarantee that a collection of classes in the subgraph extend (uniquely) to a cartesian system.

Definition 7.3.1. Let $\mathcal{X}$ be a subgraph of the graph of cartesian Selmer structures in definition 7.2.6. A *partial cartesian system* for $\mathcal{X}$ is an element of the inverse limit

$$\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}) = \varprojlim_{\mathcal{F}\in\mathcal{X}} \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

Therefore, a partial cartesian system is a collection of classes $\varepsilon_{\mathcal{F}}$ for every $\mathcal{F} \in \mathcal{X}$ such that, for every arrow $\mathcal{G} \to \mathcal{F}$ in $\mathcal{X}$, $\phi_{\mathcal{G},\mathcal{F}}(\varepsilon_{\mathcal{G}}) = \varepsilon_{\mathcal{F}}$.

Remark 7.3.2. For every subgraph $\mathcal{X} \subset \mathfrak{C}$, there is a canonical restriction map

$$\mathrm{res}: \ \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$$

Remark 7.3.3. Assume that $\mathcal{X}$ contains a Selmer structure $\mathcal{F}$ satisfying that

$$\mathrm{rank}_R \, \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee = 0$$

Then the restriction map in Remark 7.3.2 is injective. Indeed, for such Selmer structures, Theorem 7.2.21 implies that $\xi_{\mathcal{F}} \neq 0$ whenever ξ is a primitive cartesian system. Since $\mathrm{CART}_{\mathcal{F}}(\mathfrak{C})$ and $\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ are free R-modules of rank 1, then $\xi_{\mathcal{F}} \neq 0$ for any cartesian system $\xi \in \mathrm{CART}_{\mathcal{F}}(\mathfrak{C})$. Therefore, the map in Remark 7.3.2 is injective.

Notation 7.3.4. Assume that the restriction map to $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$ is injective. We then identify $\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \subset \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$ as the image of the restriction map. Those systems are partial cartesian systems that can be uniquely extended to a total one.

The goal of this section is to compute the module $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$ for certain subgraphs $\mathcal{X}$. For that reason, it is useful to express $\mathcal{X}$ as the union of the following subgraphs.

Definition 7.3.5. Let $\mathcal{X} \subset \mathfrak{C}$ be a subgraph. For every $i \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{X}^{(i)}$ the subgraph of $\mathcal{X}$ whose vertices are the Selmer structures $\mathcal{F}$ in G such that

$$\mathrm{rank}_R\, \mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \leq i$$

In an abuse of notation, we will call *corank of the Selmer structure* $\mathcal{F}$ the corank of the dual Selmer group $\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee$.

We will show that, for a non-trivial partial cartesian system $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$, the class $\xi_{\mathcal{F}} \in \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ does not vanish for any $\mathcal{F} \in \mathcal{X}^{(0)}$. Otherwise, it is expected that $\xi_{\mathcal{F}} = 0$ whenever $\mathcal{F} \notin \mathcal{X}^{(0)}$. Although it is not a general result, we will prove it under certain assumptions satisfied by all the cases of interest.

7.3.1 Zero dimensional partial cartesian systems

Fix a subgraph $\mathcal{X} \subset \mathfrak{C}$. In order to compute $\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$, we first study the partial cartesian systems in $\mathcal{X}^{(0)}$.

For the Selmer structures in $\mathcal{X}^{(0)}$, there is a weaker version of Proposition 7.2.18, but enough to control the cartesian systems at those vertices.

Lemma 7.3.6. Let $\mathcal{F} \in \mathcal{X}^{(0)}$. Then $\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ is a free, cyclic R-module.

Proof. Analogous to Lemma 7.2.16 □

The projection map in Proposition 7.2.18 is no longer an isomorphism when $\mathcal{F} \in \mathcal{X}^{(0)}$ is not a core structure. However, we can construct a control isomorphism by considering a suitable submodule of $\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$.

Definition 7.3.7. Let $\mathcal{F} \in \mathcal{X}^{(0)}$ be a cartesian Selmer structure. When R is a discrete valuation ring, we define the *defect* of $\mathcal{F}$ as

$$\delta(\mathcal{F}) := \mathrm{Fitt}^0_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

When R is the Iwasawa algebra, $\delta(\mathcal{F})$ is defined as the unique principal ideal such that

$$\delta(\mathcal{F}) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Remark 7.3.8. In the case where R is the Iwasawa algebra, for all Selmer structures $\mathcal{F}$ whose propagation to $\mathbf{T}/(X)$ is also cartesian, Theorem 6.4.3 implies that

$$\delta(\mathcal{F}) = \mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\right)$$

Proposition 7.3.9. Let $\mathcal{F} \leq \mathcal{G}$ be two cartesian Selmer structures in $\mathcal{X}^{(0)}$. The map $\Phi_{\mathcal{G},\mathcal{F}}$ induces an isomorphism:

$$\Phi_{\mathcal{G},\mathcal{F}}: \ \delta(\mathcal{G})\mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \to \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

Proof. By Proposition 7.2.15, we can find a core Selmer structure $\mathcal{H}$ such that $\mathcal{G} \leq \mathcal{H}$. By Propositions B.4.13 and B.4.14, there are isomorphisms

$$\Phi_{\mathcal{H},\mathcal{G}}: \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}) \to \delta(\mathcal{G})\mathbf{X}_{\mathcal{F}_0}(\mathcal{G})$$
$$\Phi_{\mathcal{H},\mathcal{F}}: \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}) \to \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

By Proposition 7.2.11, we know that

$$\Phi_{\mathcal{G},\mathcal{F}} = \Phi_{\mathcal{H},\mathcal{G}} \circ \Phi^{-1}_{\mathcal{H},\mathcal{F}}: \ \delta(\mathcal{G})\mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \to \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

is an isomorphism. □

By linearity, the map $\Phi_{\mathcal{G},\mathcal{F}}$ can be extended to a map defined on all $\mathbf{X}_{\mathcal{F}_0}(\mathcal{G})$.

Corollary 7.3.10. Let $\mathcal{F} \leq \mathcal{G}$ be two cartesian Selmer structures. By Proposition 4.3.11, there is a surjection

$$\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee \twoheadrightarrow \mathbf{H}^1_{\mathcal{G}^*}(K, T^*)^\vee$$

The computations of Fitting ideals in Propositions B.2.6 and B.2.9 imply that $\delta(\mathcal{F}) \subset \delta(\mathcal{G})$. Since both are principal ideals, we can find a principal ideal $\delta_{\mathcal{G}/\mathcal{F}}$ such that $\delta(\mathcal{G})\delta_{\mathcal{G}/\mathcal{F}} = \delta(\mathcal{F})$. Then $\Phi_{\mathcal{G},\mathcal{F}}$ can be extended by linearity to an isomorphism

$$\Phi_{\mathcal{G},\mathcal{F}}: \ \mathbf{X}_{\mathcal{F}_0}(\mathcal{G}) \to \delta_{\mathcal{G}/\mathcal{F}}\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

Note that, if $\mathcal{G} \to \mathcal{F}$ is an arrow in $\mathcal{X}^{(0)}$, then every partial cartesian system $\xi \in \mathrm{CART}_{\mathcal{F}_0}\left(\mathcal{X}^{(0)}\right)$ satisfies that $\Phi_{\mathcal{G},\mathcal{F}}(\xi_G) = \xi_{\mathcal{F}}$.

The existence of these isomorphisms implies that every structure controls the cartesian systems in $\mathcal{X}^{(0)}$.

Corollary 7.3.11. Let $\mathcal{X}^{(0)}$ be a connected cartesian subgraph and let $\mathcal{F} \in \mathcal{X}^{(0)}$. Then the projection map

$$\mathrm{CART}_{\mathcal{F}_0}\big(\mathcal{X}^{(0)}\big) \to \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

is injective.

Our next goal is to determine which partial cartesian systems in $\mathcal{X}^{(0)}$ can be extended uniquely to a total cartesian system.

Corollary 7.3.12. For every $\mathcal{F} \in \mathcal{X}^{(0)}$ there is an isomorphism

$$\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \to \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

Proof. Let $\mathcal{G} \in \mathfrak{C}$ be a core Selmer structure such that $\mathcal{F} \leq \mathcal{G}$. By Proposition 7.2.17, there are isomorphisms

$$\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \xrightarrow{\sim} \mathbf{X}_{\mathcal{F}_0(\mathcal{G})} \xrightarrow{\Phi_{\mathcal{G},\mathcal{F}}} \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0(\mathcal{F})}$$

□

This isomorphism helps to control the image of the restriction map in Remark 7.3.2.

Corollary 7.3.13. Assume that $\mathcal{X}^{(0)}$ is non-empty. Then the canonical restriction map

$$\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$$

is injective.

Proof. It follows from Corollary 7.3.12 and the commutativity of the diagram:

$$\begin{tikzcd} \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C}) \arrow[r] \arrow[dr] & \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)}) \arrow[d] \\ & \mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \end{tikzcd}$$

By Corollary 7.3.12, the diagonal map is injective. Therefore, the restriction map is also injective. □

It is possible to use Corollary 7.3.12 to compute the image of the restriction map. At this point, we assume that $\mathcal{X}^{(0)}$ is connected. The general case can be studied by studying each connected component individually.

Corollary 7.3.14. Assume that $\mathcal{X}^{(0)}$ is connected and let $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$ be a cartesian system satisfying that

$$\xi_{\mathcal{F}} \in \delta(\mathcal{F})\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

for one Selmer structure in $\mathcal{X}^{(0)}$. Then $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$ is the restriction of a total cartesian system.

Corollary 7.3.15. Assume that $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$ is a partial cartesian system. For any $\mathcal{F} \in \mathcal{X}^{(0)}$ and $\alpha \in \delta(\mathcal{F})$, $\alpha\xi$ is the restriction of a total cartesian system.

This implies that the defects $\delta(\mathcal{F})$ annihilate the quotient

$$\frac{\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})}{\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})}$$

This cokernel can be controlled by the defect of all Selmer structures in $\mathcal{X}^{(0)}$.

Definition 7.3.16. Let $\mathcal{X}^{(0)} \subset \mathfrak{C}^{(0)}$ be a subgraph of the cartesian Selmer structures consisting only of Selmer structures whose dual Selmer group has corank 0. We define the *defect* of $\mathcal{X}^{(0)}$ as the

$$\delta\big(\mathcal{X}^{(0)}\big) := \sum_{\mathcal{F} \in \mathcal{X}^{(0)}} \delta(\mathcal{F})$$

Remark 7.3.17. $\delta\big(\mathcal{X}^{(0)}\big) = 1$ if and only if $\mathcal{X}^{(0)}$ contains a core Selmer structure.

Proposition 7.3.18. Assume $\mathcal{X}^{(0)}$ is connected. Then $\delta(\mathcal{X})$ annihilates the quotient

$$\frac{\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})}{\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})}$$

Proof. By Corollary 7.3.15, $\delta(\mathcal{F})$ annihilates the quotient for any $\mathcal{F} \in \mathcal{X}^{(0)}$. Hence, $\delta(\mathcal{X}^{(0)})$ is also contained in the annihilator. $\square$

Theorem 7.3.19. Let $\mathcal{X}^{(0)}$ be a connected subgraph of $\mathfrak{C}^{(0)}$ whose defect is denoted by $\delta(\mathcal{X}^{(0)})$ and let $\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$ be a partial cartesian system. For each cartesian Selmer structure $\mathcal{F} \in \mathcal{X}^{(0)}$, we have that

$$\mathrm{Ind}(\xi_{\mathcal{F}})\delta(\mathcal{X}^{(0)}) \subset \mathrm{Fitt}^0_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Proof. By Proposition 7.3.18,

$$\delta(\mathcal{X}^{(0)})\xi \in \mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})$$

Then the theorem follows from Theorem 7.2.21. $\square$

We finalise this section describing the module $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$. When R is a discrete valuation ring, it is isomorphic to a free module of rank one, while in the case where R is the Iwasawa algebra, we can only say it is pseudo-isomorphic to a free Iwasawa module.

Theorem 7.3.20. Let $\mathcal{X}^{(0)} \subset \mathfrak{C}^{(0)}$ be a non-empty, connected cartesian subgraph. Then $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$ is a free R-module of rank one.

Proof. When R is a discrete valuation ring, it follows from Lemma 7.3.6 and Corollary 7.3.11.

In the case where R is the Iwasawa algebra, those results imply that $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$ is contained with finite index in a free module.

For every $\mathcal{F} \in \mathcal{X}^{(0)}$ denote by $\mathbf{Z}(\mathcal{F})$ the image of the projection map

$$\mathrm{CART}_{\mathcal{F}_0}\big(X^{(0)}\big) \hookrightarrow \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$$

and by $\mathbf{Y}(\mathcal{F})$ the unique cyclic submodule of $\mathbf{X}(\mathcal{F})$ satisfying that $\mathbf{Z}(\mathcal{F}) \approx \mathbf{Y}(\mathcal{F})$.

By construction, for each arrow $\mathcal{G} \to \mathcal{F}$ in $\mathcal{X}^{(0)}$, we have that $\Phi_{\mathcal{G},\mathcal{F}}(\mathbf{Z}(\mathcal{G})) = \mathbf{Z}(\mathcal{F})$. That implies that $\Phi_{\mathcal{G},\mathcal{F}}(\mathbf{Y}(\mathcal{G})) = \mathbf{Y}(\mathcal{F})$, so every element in $\mathbf{Y}(\mathcal{F})$ can be obtained as the projection of a partial cartesian system. It shows that $\mathbf{Y}(\mathcal{F}) = \mathbf{Z}(\mathcal{F})$ for every $\mathcal{F} \in \mathbf{X}^{(0)}$, so $\mathrm{CART}_{\mathcal{F}}\big(\mathcal{X}^{(0)}\big)$ is a cyclic module. $\square$

Definition 7.3.21. A partial cartesian system in $\mathrm{CART}_{\mathcal{F}}(\mathcal{X}^{(0)})$ is called *primitive* if it generates $\mathrm{CART}_{\mathcal{F}}(\mathcal{X}^{(0)})$ as an R-module.

In addition, we want to compare the partial cartesian sytems to the total ones.

Definition 7.3.22. Let $\mathcal{X}^{(0)}$ be a partial cartesian subgraph. We call the *principal defect* $\widetilde{\delta}\big(\mathcal{X}^{(0)}\big)$ as the unique principal ideal of R pseudo-equal to $\delta(\mathcal{F})$. Of course, the defect and principal defect coincide when R is a discrete valuation ring.

Theorem 7.3.23. There is an isomorphism

$$\frac{\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})}{\mathrm{CART}_{\mathcal{F}_0}(\mathfrak{C})} \cong \frac{R}{\widetilde{\delta}(\mathcal{F})}$$

Proof. Let α be a generator of $\widetilde{\delta}(\mathcal{F})$ and let ξ be a primitive cartesian system. For every $\mathcal{F} \in \mathcal{X}^{(0)}$, $\delta(\mathcal{F}) \subset \widetilde{\delta}\big(\mathcal{X}^{(0)}\big)$, so Corollary 7.3.12 implies the existence of an element $\lambda_{\mathcal{F}} \in \mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$ such that $\alpha\lambda_{\mathcal{F}} = \xi_{\mathcal{F}}$. It is clear that the collection

$$\{\lambda_{\mathcal{F}} :\ \mathcal{F} \in \mathcal{X}^{(0)}\} \in \mathrm{CART}_{\mathcal{F}_0}\big(\mathcal{X}^{(0)}\big)$$

Proposition 7.3.18 implies that it is primitive, which concludes this proof. □

From Theorem 7.2.21, we can conclude the following

Corollary 7.3.24. Let $\xi \in \mathrm{CART}_{\mathcal{F}_0}\big(\mathcal{X}^{(0)}\big)$ be a primitive partial cartesian system and let $\mathcal{F}$ be a cartesian Selmer structure. If R is a discrete valuation ring, then

$$\delta\big(\mathcal{X}^{(0)}\big)\,\mathrm{ind}(\xi_{\mathcal{F}}) = \mathrm{Fitt}^0_R\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

Otherwise, when R is the Iwasawa algebra, the above is only true up to pseudo-equality:

$$\widetilde{\delta}\big(\mathcal{X}^{(0)}\big)\,\mathrm{ind}(\xi_{\mathcal{F}}) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)^\vee\big)$$

7.3.2 Higher dimensional partial cartesian systems

We have studied the partial cartesian system in $\mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$. Under certain assumptions on $\mathcal{X}$, it is enough to compute $\mathrm{CART}_{\mathcal{F}}(\mathcal{X})$ since

$$\xi_{\mathcal{F}} = 0\ \forall \xi \in \mathrm{CART}_{\mathcal{F}}(\mathcal{X}),\ \forall \mathcal{F} \in \mathcal{X} \setminus \mathcal{X}^{(0)} \tag{7.3}$$

Note that, by Theorem 7.2.21, that is always the case when ξ is the restriction of a total cartesian system in $\mathrm{CART}_{\mathcal{F}}(\mathfrak{C})$.

However, (7.3) only holds true for some well behaved graphs $\mathcal{X}$ and, in more generality, there might be cartesian systems not vanishing at structures of higher corank. We are not interested in those systems, since they have no control over the Selmer groups. Therefore, we will work on establishing conditions on the graph $\mathcal{X}$ that ensure that this situation cannot happen.

We can show that, when there is an edge from $\mathcal{X}^{(i)}$ to $\mathcal{X}^{(i+1)}$, the partial cartesian systems vanish. That is written precisely with the definition of cascades.

Definition 7.3.25. Let $\mathcal{Y} \subset \mathcal{X}$ be subgraphs of $\mathfrak{C}$. We define the *cascade* $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ as the graph whose Selmer structures are the $\mathcal{F} \in \mathcal{X}$ such that there is another Selmer structure $\mathcal{G} \in \mathcal{Y}$ such that there is an edge $\mathcal{G} \to \mathcal{F}$ in $\mathcal{X}$.

The idea is that, when a Selmer structure is in the cascade of Selmer structures of smaller corank, then every partial cartesian system vanishes at $\mathbf{X}_{\mathcal{F}_0}(\mathcal{F})$.

Proposition 7.3.26. Let $\mathcal{F} \in \widehat{\mathcal{X}^{(i)}} \setminus \mathcal{X}^{(i)}$ for some non-negative integer i. Then $\varepsilon_{\mathcal{F}} = 0$ for all $\varepsilon \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X})$.

Proof. By Definition 7.3.25, there is a cartesian Selmer structure $\mathcal{G} \in \mathcal{X}^{(i)}$ such that $\mathcal{F} \leq \mathcal{G}$ and the arrow $\mathcal{G} \to \mathcal{F}$ is in $\mathcal{X}$. By Proposition 4.3.11, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{H}^1_{\mathcal{F}}(K,T) \arrow[r] & \mathbf{H}^1_{\mathcal{G}}(K,T) \arrow[r] & \mathcal{L}_{\mathcal{G}/\mathcal{F}} \arrow[dll, out=0, in=180] & \\
& \mathbf{H}^1_{\mathcal{F}^*}(K,T^*)^\vee \arrow[r] & \mathbf{H}^1_{\mathcal{G}^*}(K,T^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{F} \notin \mathcal{X}^{(i)}$, the image of the map

$$\mathbf{H}^1_{\mathcal{G}}(K,T) \to \mathcal{L}_{\mathcal{G}/\mathcal{F}}$$

has rank smaller than $\chi(\mathcal{G}) - \chi(\mathcal{F})$. That implies that the map

$$\det(\mathcal{L}^+_{\mathcal{G}/\mathcal{F}}) \to \bigwedge^{\chi(\mathcal{G})-\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{G}/\mathcal{F}}(K,T)^+$$

is the zero-map. Hence the map

$$\Phi_{\mathcal{G},\mathcal{F}} : \bigcap^{\chi(\mathcal{G})} \mathbf{H}^1_{\mathcal{G}}(K,T) \to \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}}(K,T)$$

ed from Propositions B.4.7 and B.4.8, is also the zero map. Hence

$$\xi_{\mathcal{F}} = \Phi_{\mathcal{G},\mathcal{F}}(\xi_{\mathcal{F}}) = 0 \; \forall \varepsilon \in \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}) \qquad \square$$

In order to extend the above result to Selmer structures which are not in the cascades of smaller coranks, we can use the following result.

Proposition 7.3.27. Let $\mathcal{F} \in \mathcal{X} \setminus \mathcal{X}^{(i)}$ for some non-negative integer i. Assume there are Selmer structures $\mathcal{H}_1, \ldots, \mathcal{H}_s \in \widehat{\mathcal{X}^{(i)}}$ with arrows $\mathcal{F} \to \mathcal{H}_1, \ldots, \mathcal{F} \to \mathcal{H}_s$ such that the map

$$\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \to \bigoplus_{i=1}^{s} \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}_i)$$

is injective. If $\varepsilon \in \mathrm{CART}_{\mathcal{F}_0}(G)$, then $\varepsilon_{\mathcal{F}} = 0$.

Proof. For every $j = 1, \ldots, s$, we have that $\mathcal{H}_j \leq \mathcal{F}$, so there is a surjective map

$$\mathbf{H}^1_{(\mathcal{H}_j)^*}(K, T^*)^\vee \twoheadrightarrow \mathbf{H}^1_{(\mathcal{F})^*}(K, T^*)^\vee$$

Therefore $\mathcal{H}_j \notin \mathcal{X}^{(i)}$, so Proposition 7.3.26 implies that $\varepsilon_{\mathcal{H}_j} = 0$. Then,

$$\varepsilon_{\mathcal{F}} \in \ker\Big(\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \to \bigoplus_{i=1}^{s} \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}_i)\Big) = \{0\} \qquad \square$$

Remark 7.3.28. The previous result is based on the ideas used in §6.3 to control the module of Kolyvagin systems.

Corollary 7.3.29. Assume $\mathcal{X} \subset \mathfrak{C}$ is a subgraph satisfying that for every Selmer structure of corank $i > 0$, there are are Selmer structures $\mathcal{H}_1, \ldots, \mathcal{H}_s \in \widehat{\mathcal{X}^{(i-1)}}$ such that there is an arrow $\mathcal{F} \to \mathcal{H}_j$ in $\mathcal{X}$ for all $j = 1, \ldots, s$ and the map

$$\mathbf{X}_{\mathcal{F}_0}(\mathcal{F}) \to \bigoplus_{i=1}^{s} \mathbf{X}_{\mathcal{F}_0}(\mathcal{H}_i)$$

is injective. Then the restriction map

$$\mathrm{res}: \ \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}) \to \mathrm{CART}_{\mathcal{F}_0}(\mathcal{X}^{(0)})$$

is an isomorphism and the inverse map is given by the extension of the system by 0.

7.4 Examples of partial cartesian systems

7.4.1 Stark systems

In §6.2, we have defined Stark systems as a collection of elements

$$\varepsilon_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F})+\nu(\mathfrak{n})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, T)$$

On the one hand, note that $\chi(\mathcal{F}) + \nu(\mathfrak{n}) = \chi(\mathcal{F}^{\mathfrak{n}})$. On the other hand, the construction of Stark systems was based on choices, for all $\mathfrak{u} \in \mathcal{P}$, of generators

$$\psi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T)^+$$

and an order relation on $\mathcal{P}$. Taking this choices into account, we can consider the tensor product

$$\xi_{\mathcal{F}^n} := \varepsilon_n \otimes \bigotimes_{\mathfrak{u}|\mathfrak{n}} \psi_{\mathfrak{u}} \in \bigcap^{\chi(\mathcal{F}^{\mathfrak{n}})} \mathbf{H}^1_{\mathcal{F}^{\mathfrak{n}}}(K, T) \otimes \det(\mathcal{L}^+_{\mathcal{F}(\mathfrak{n})/\mathcal{F}}) = \mathbf{X}_{\mathcal{F}}(\mathcal{F}^{\mathfrak{n}})$$

The stark systems relation shows that ξ is a partial cartesian system on the graph

$$\mathcal{X} = \{\mathcal{F}^{\mathfrak{n}} : \ \mathfrak{n} \in \mathcal{N}\}$$

and arrows $\mathcal{F}^{\mathfrak{r}} \to \mathcal{F}^{\mathfrak{n}}$ whenever $\mathfrak{n} \mid \mathfrak{r}$.

There is then a canonical restriction map

$$\mathrm{res}: \ \mathrm{CART}_{\mathcal{F}}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}}(\mathcal{X}) = \mathbf{SS}(\mathcal{F})$$

We will show that the restriction map is in fact an isomorphism. Note that Proposition 6.1.6 implies that

$$\mathcal{X} = \widehat{\mathcal{X}^{(0)}}$$

Hence Proposition 7.3.26 implies that

$$\mathrm{CART}_{\mathcal{F}}(\mathcal{X}) = \mathrm{CART}_{\mathcal{F}}(\mathcal{X}^{(0)})$$

Note that $\mathcal{X}^{(0)}$ is connected and it contains core Selmer structures by Proposition 6.1.6, so $\delta\big(\mathcal{X}^{(0)}\big) = 1$ by Remark 7.3.17. That means that the restriction map

$$\mathrm{res}: \ \mathrm{CART}_{\mathcal{F}}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}}(\mathcal{X}^{(0)})$$

is an isomorphism. Therefore, there is a canonical identification between total cartesian and Stark systems:

$$\mathrm{CART}_{\mathcal{F}}(\mathfrak{C}) = \mathbf{SS}_(\mathcal{F})$$

7.4.2 Kolyvagin systems

Assume that $\mathcal{F}$ is a cartesian Selmer structure of core rank $\chi(\mathcal{F}) \geq 1$. In Definition 6.3.1, we have defined Kolyvagin systems as a collection of classes

$$\kappa_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T)$$

for $\mathfrak{n} \in \mathcal{N}$. However, the relation between $\kappa_{\mathfrak{n}}$ and $\kappa_{\mathfrak{n}\mathfrak{u}}$ involves the choice of a finite-singular map. In particular, it requires the choice of a generator of $\mathcal{G}_{\mathfrak{u}}$ (see Remark 5.1.35). In some places of the literature (see, for example, [MR04],[BSS25]), Kolyvagin systems are defined as a collection of classes

$$\kappa_{\mathfrak{n}} \in \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \otimes \bigotimes_{\mathfrak{u}|\mathfrak{n}} \mathcal{G}_{\mathfrak{u}}$$

so it is natural to consider a relation between $\kappa_{\mathfrak{n}}$ and $\kappa_{\mathfrak{n}\mathfrak{u}}$ given by the finite-singular map in the form of Proposition 5.1.34.

Proposition 5.1.34 also constructs, for every $\mathfrak{u} \in \mathcal{P}$, a finite-singular map $\phi^{\mathrm{fs}}_{\mathfrak{u}}$, whose inverse can be explicitly written as

$$(\phi^{\mathrm{fs}}_{\mathfrak{u}})^{-1}: \ \mathcal{G}_{\mathfrak{u}} \otimes \mathbf{H}^1_{\mathrm{s}}(K\mathfrak{u}, T) \to \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$$

By the universal property of the tensor product, there is another map

$$\mathcal{G}_{\mathfrak{u}} \to \mathrm{Hom}(\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T), \mathbf{H}^1_{\mathrm{f}}(K, T)) = \mathbf{H}^1_{\mathrm{s}}(K, T)^+ \otimes \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)$$

The last term can be expressed as a determinant, in order to obtain a map

$$\mathcal{G}_{\mathfrak{u}} \to \det\big(\mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T)^+ \oplus \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)\big)$$

Combining this for all ultraprime divisors of $\mathfrak{n}$, we obtain a map

$$\bigotimes_{\mathfrak{u}|\mathfrak{n}} \mathcal{G}_{\mathfrak{u}} \to \det(\mathcal{L}^+_{\mathcal{F}(\mathfrak{n})/\mathcal{F}})$$

Under this identification, Kolyvagin systems can be seen as collections of classes

$$\kappa_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \otimes \det(\mathcal{L}^+_{\mathcal{F}(\mathfrak{n})/\mathcal{F}}) = \mathbf{X}_{\mathcal{F}}(\mathcal{F}(\mathfrak{n}))$$

Therefore, Kolyvagin systems can be seen as partial cartesian systems on the graph $\mathcal{X}$ with arrows given by

$$\mathcal{F}(\mathfrak{n}) \to \mathcal{F}_{\mathfrak{u}}(\mathfrak{n}), \quad \mathcal{F}(\mathfrak{n}\mathfrak{u}) \to \mathcal{F}_{\mathfrak{u}}(\mathfrak{n}), \quad \forall \mathfrak{n} \in \mathcal{N}, \ \forall \mathfrak{u} \in \mathcal{P}, \ \mathfrak{u} \nmid \mathfrak{n}$$

where $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{u} \in \mathcal{P}$ is an ultraprime not dividing $\mathfrak{n}$.

If we choose a generator $\phi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathrm{f}}(K_{\mathfrak{u}}, T)^+$ as in Chapter 6, we can see the following tensor product as an element in

$$\kappa_{\mathfrak{n}} \otimes \phi_{\mathfrak{u}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \otimes \det(\mathcal{L}^+_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})/\mathcal{F}})$$

Then, we can compute that

$$\left(\alpha_{\mathfrak{n},\mathfrak{u}} \otimes \mathrm{Id}_{\det\left(\mathcal{L}^+_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})/\mathcal{F}}\right)}\right)(\kappa_{\mathfrak{n}} \otimes \phi_{\mathfrak{u}}) = \Phi_{\mathcal{F}(\mathfrak{n}),\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(\kappa_n) \in \mathbf{X}_{\mathcal{F}}(\mathcal{F}_{\mathfrak{u}}(\mathfrak{n}))$$

where $\alpha_{\mathfrak{n},\mathfrak{u}}$ is the map defined in §6.3.

Similarly, we also have constructed in Chapter 6 a generator $\psi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, \mathbf{T})$ satisfying that $\psi_{\mathfrak{u}} \in \mathbf{H}^1_{\mathrm{s}}(K_{\mathfrak{u}}, T)^+$ such that $\psi_{\mathfrak{u}} \circ \phi^{\mathrm{fs}}_{\mathfrak{u}} = \phi_{\mathfrak{u}}$. We can also consider the tensor products

$$\kappa_{\mathfrak{n}\mathfrak{u}} \otimes \psi_{\mathfrak{u}} \in \bigcap^{\chi(\mathcal{F})} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n}\mathfrak{u})}(K, T) \otimes \det(\mathcal{L}^+_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})/\mathcal{F}})$$

We can further compute the following equality:

$$\left(\beta_{\mathfrak{n},\mathfrak{u}} \otimes \mathrm{Id}_{\det\left(\mathcal{L}^+_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})/\mathcal{F}}\right)}\right)(\kappa_{\mathfrak{n}\mathfrak{u}} \otimes \psi_{\mathfrak{u}}) = \Phi_{\mathcal{F}(\mathfrak{n}\mathfrak{u}),\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}(\kappa_{\mathfrak{n}\mathfrak{u}}) \in \mathbf{X}_{\mathcal{F}}(\mathcal{F}_{\mathfrak{u}}(\mathfrak{n}))$$

where $\beta_{\mathfrak{n},\mathfrak{u}}$ is the map defined in §6.3. The Kolyvagin system relation implies that the two elements constructed in $\mathbf{X}_{\mathcal{F}}(\mathcal{F}_{\mathfrak{u}}(\mathfrak{n}))$ coincide, and are defined to be the element in $\xi_{\mathcal{F}_{\mathfrak{u}}(\mathfrak{n})}$ in our partial cartesian system.

This construction shows the equivalence between the Kolyvagin systems and partial cartesian systems in $\mathcal{X}$.

Theorem 7.4.1. There is a canonical isomorphism

$$\mathbf{KS}(\mathcal{F}) = \mathrm{CART}_{\mathcal{F}}(\mathcal{X})$$

In addition, Kolyvagin systems extend uniquely to a total cartesian system.

Theorem 7.4.2. The canonical restriction map is an isomorphism

$$\mathrm{CART}_{\mathcal{F}}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}}(\mathcal{X}) = \mathrm{KS}(\mathcal{F})$$

Proof. It follows from Theorem 6.3.5 and Proposition 7.2.17. □

Remark 7.4.3. The argument behind the control of the Kolyvagin systems in §6.3 is analogous to the arguments in §7.3. Indeed, the argument in §6.3 starts by showing that $\mathcal{X}^{(0)}$ is connected and then uses an analogous argument to Corollary 7.3.29 to show that the Kolyvagin class $\kappa_{\mathfrak{n}}$ vanishes whenever the corank of $\mathbf{H}^1_{\mathcal{F}^*}(K, \mathbf{T}^*)$ is non-zero.

Remark 7.4.4. It is expected that the theory of cartesian systems will contribute to the study of Kolyvagin systems when Assumption (H2) does not hold. This assumption is only required to guarantee the existence of core vertices, which will not exist in more general cases. However, one could show that $\mathcal{X}^{(0)}$ is still non-empty in this case, so Corollary 7.3.24 could be used to control the Kolyvagin systems. In this case, the defect of the graph is expected to measure error terms arising from the existence of non-trivial Tamagawa numbers and that play a role in Kato's formulation of the Iwasawa main conjecture. That theory will intersect with recent work in [BB25].

Remark 7.4.5. Cartesian systems can be used as a unified theory of Kolyvagin systems, containing both cyclotomic Kolyvagin systems (those appearing in this thesis) and anticyclotomic Kolyvagin systems (arising from Heegner points).

7.4.3 Kolyvagin systems of rank zero

Definition 5.2.4 only defines Kolyvagin systems in the case where the core rank $\chi(\mathcal{F})$ is positive. The reason behind this is that the Kolyvagin systems relations are compared in an exterior bidual of rank $\chi(\mathcal{F}) - 1$, which has only been constructed when $\chi(\mathcal{F}) = 1$.

However, when $\chi(\mathcal{F}) = 0$, all the structures $\mathcal{F}(\mathfrak{n})$ also have core rank 0 when $\mathfrak{n} \in \mathcal{N}$. Then, it is possible to construct Kolyvagin classes

$$\kappa_{\mathfrak{n}} \in \bigcap^{\chi(\mathcal{F}(\mathfrak{n}))} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K, T) \cong R$$

but it is not possible to define Kolyvagin system relations.

The theory of cartesian systems allows a new approach to construct Kolyvagin systems of rank 0. Consider the cartesian subgraph $\mathcal{X}$ defined as

$$\mathcal{X}_{\mathcal{F}} = \{\mathcal{F}(\mathfrak{n}) : \mathfrak{n} \in \mathcal{N}\}$$

with no arrows connecting them. With this definition, there is an isomorphism

$$\mathrm{CART}_{\mathcal{F}}(\mathcal{X}) \cong \prod_{\mathfrak{n}\in\mathcal{N}} R$$

However, there is a subspace of them that control the Selmer groups: those arising from a total cartesian system.

Definition 7.4.6. Let $\mathcal{F}$ be a cartesian Selmer structure. The set of Kolyvagin systems with $\mathcal{F}$ is defined as

$$\mathbf{KS}(\mathcal{F}) := \mathrm{Im}\left(\mathrm{CART}_{\mathcal{F}}(\mathfrak{C}) \to \mathrm{CART}_{\mathcal{F}}(\mathcal{X}_{\mathcal{F}})\right) \subset \prod_{\mathfrak{n}\in\mathcal{N}} \mathbf{H}^1_{\mathcal{F}(\mathfrak{n})}(K,T)$$

Remark 7.4.7. Note that, by Theorem 7.4.2, Definition 7.4.6 is equivalent to Definition 5.2.4 when $\chi(\mathcal{F}) > 0$.

By Lemma 7.2.16 and Theorem 7.2.21, we obtain the following result.

Corollary 7.4.8. Let $\mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{F}) = 0$. The module of Kolyvagin systems $\mathbf{KS}(\mathcal{F})$ is free of rank one and any generator κ satisfies that

$$\mathrm{ind}(\kappa_{kn}) \approx \mathrm{Fitt}^0_\Lambda\left(\mathbf{H}^1_{\mathcal{F}^*(\mathfrak{n})}(K,T)^\vee\right)$$

Remark 7.4.9. Let $\mathcal{G} \geq \mathcal{F}$ be a cartesian Selmer structure of core rank $\chi(\mathcal{G}) = 1$. The patching of the quantities $\delta_n(\kappa, \mathcal{F})$ constructed in Definition 3.3.5 form a Kolyvagin system of rank 0 for the Selmer structure $\mathcal{F}$, which will be primitive whenever κ is primitive.

Remark 7.4.10. One could generalise the definition of 5.2.22 for rank zero Kolyvagin systems in $\mathbf{KS}(\mathcal{F})$. They would coincide with the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F})$, constructed in Definition 5.4.2, for any Selmer structure $\mathcal{G} \geq \mathcal{F}$ of core rank $\chi(\mathcal{G}) = 1$ and some Kolyvagin system $\kappa \in \mathrm{KS}(\mathcal{G})$. Hence, the rank 0 theta ideals will be related to the Fitting ideals of the Selmer group as in Theorems 5.4.5 and 5.4.6.

Remark 7.4.11. There has been an alternative approach to construct Kolyvagin systems of rank 0 in [Sak21]. This approach considers the product of the rank one Kolyvagin systems for all the Selmer structures $\mathcal{F}^\ell$ where ℓ is a Kolyvagin prime. Our approach constructs directly the quantities δ_n appearing in that approach.

Remark 7.4.12. It is an interesting question to give a characterization of Kolyvagin systems of rank 0, similar to the definitions of Kolyvagin systems for Selmer structures of positive core rank.

Part IV

Elliptic Curves

Chapter 8

Selmer group of an elliptic curve

8.1 Main results

The aim of this section is to apply Theorems 3.3.7 and 3.4.1 to compute the Galois structure of the classical Selmer group of an elliptic curve (defined over $\mathbb{Q}$) over certain abelian extensions. Hence, we give an explicit computations of the ideals $\Theta_i^{(0)}(\kappa, \mathcal{F}_{\mathrm{BK}})$ from Definition 3.3.6, where κ is the Kolyvagin system obtained from Kato's Euler system, which was constructed in A.3.2 and $\mathcal{F}_{\mathrm{BK}}$ is the Bloch-Kato Selmer structure defined in Definition 2.3.4. The results in this chapter were already included in [Ang25, Theorem 1.5].

8.1.1 Assumptions

Throughout this chapter, let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$ and let p be a prime number. Denote by N the conductor of E and by T the Tate module of E.

Assume the following hypothesis.

Assumption (EC). The elliptic curve $E/\mathbb{Q}$ and the prime p satisfy the following

- (E0) The prime p is at least 5.
- (E1) The homomorphism $\rho: \ G_{\mathbb{Q}} \to \mathrm{Aut}(T)$ induced by the Galois action is surjective.
- (E2) Either the Manin constant c_0 or c_1 is prime to p (see §8.1.3)

Assumption (E1) is necessary to ensure that T satisfies the Assumption (BI), using Remark 2.1.4. Assumption (E2) is necessary to guarantee the integrality of the modular symbols, which will be defined below in §8.1.3.

Although we need the elliptic curve E to be defined over $\mathbb{Q}$ in order to apply the modularity theorem, required for the definition of modular symbols, we will study the properties of the Mordell-Weil group over a finite abelian extension $K/\mathbb{Q}$ satisfying the following hypotheses:

Assumption (Ab). The abelian extension $K/\mathbb{Q}$ satisfies the following:

- (K1) $K/\mathbb{Q}$ is unramified at p and at every prime at which E has bad reduction.
- (K2) The degree $[K : \mathbb{Q}]$ is prime to p.
- (K3) $E(K_{\mathfrak{p}})[p] = \{O\}$ for every prime $\mathfrak{p}$ of K above p.
- (K4) All the Tamagawa numbers of E over K are prime to p.
- (K5) For every character χ of $\mathrm{Gal}(K/\mathbb{Q})$, the Iwasawa main conjecture localised at $X\Lambda$ holds for the modular form f_χ (see Remark 8.1.17 below).

Notation 8.1.1. We denote by G the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Also, let d and c denote the degree and the conductor of $K/\mathbb{Q}$, respectively. We will also denote by $\mathcal{O}_d$ the ring of integers of $\mathbb{Q}_p(\mu_d)$ and $\Lambda_d := \Lambda \otimes \mathcal{O}_d$, where Λ is the Iwasawa algebra.

When v is a prime of K above a rational prime ℓ. Denote by $G_{v/\ell}$ to the Galois group $\mathrm{Gal}(K_v/\mathbb{Q}_\ell)$, which is canonically isomorphic to a subgroup of G.

8.1.2 The Selmer structure

Let $E/\mathbb{Q}$ be an elliptic curve and let K be a number field. There is a standard p-adic Galois representation associated with the elliptic curve, commonly known as the Tate module:

$$T_pE = \varprojlim_{n\in\mathbb{N}} E[p^n]$$

The Tate module is a free $\mathbb{Z}_p$-module of rank 2 endowed with a natural action of the Galois group $G_{\mathbb{Q}}$. This action is only ramified at p, possibly at archimedean primes and at bad reduction primes of E, by the Néron-Ogg-Shafarevich criterion (see [Sil09, Theorem 7.1]). Thus we are under Assumption (SI).

There is a classical Selmer structure defined on this Galois representation. In order to define the local conditions, consider the Kummer maps for every prime $\ell \in \mathbb{P}_K$.

$$0 \longrightarrow E(K_\ell)\widehat{\otimes}\mathbb{Z}_p \xrightarrow{\kappa_\ell} H^1(K_\ell, T_pE) \longrightarrow H^1(K_\ell, E) \longrightarrow 0$$

The image of the Kummer maps define a Selmer structure, also known as classical.

Definition 8.1.2. The classical Selmer structure on the Tate module is defined by the local conditions

$$H^1_{\mathcal{F}_{\mathrm{cl}}}(K_\ell, T) = \mathrm{Im}(\kappa_\ell) \cong E(\mathbb{Q}_\ell) \otimes \mathbb{Z}_p$$

The importance of the Selmer group is its control over the rank of the Mordell-Weil group $E(K)$. Indeed, it fits into the short exact sequence

$$0 \longrightarrow E(K)\widehat{\otimes}\mathbb{Z}_p \longrightarrow H^1_{\mathcal{F}_{\mathrm{cl}}}(K, T_pE) \longrightarrow Ш(E/K)[p^\infty] \longrightarrow 0$$

where $Ш(E/K)[p^\infty]$ is the p-primary part of the Tate-Shafarevich group. Hence, the classical Selmer group establishes an upper bound on the rank of the elliptic curve. It is conjectured that the Tate-Shafarevich group is finite, so this upper bound would become the exact rank.

In fact, the definition of the Bloch-Kato Selmer structure 2.3.4 is a generalisation of the classical Selmer structure to all Galois representations.

Proposition 8.1.3. ([BK90, §3]) The classical and the Bloch-Kato Selmer structures coincide on the Tate module of an elliptic curve.

In order to study the Galois structure of the classical Selmer group, we can apply Shapiro's lemma (see [NSW00, Proposition 1.6.4]) and study the Galois representation

$$T := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$$

Indeed, Shapiro's lemma induces an isomorphism

$$H^1(K, T) \cong H^1\left(\mathbb{Q}, \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(T)\right) \cong H^1(\mathbb{Q}, T \otimes \mathbb{Z}_p[G])$$

At this point, it is important to make a remark on the Galois action on $T \otimes \mathbb{Z}_p[G]$. In order to construct the Galois cohomology group $H^1(\mathbb{Q}, T \otimes \mathbb{Z}_p[G])$, we need $T \otimes \mathbb{Z}_p[G]$ to be endowed with a continuous action of $G_{\mathbb{Q}}$. It is given by the following formula:

$$\sigma(t \otimes x) = \sigma t \otimes \sigma x \ \forall \sigma \in G, \ \forall t \in T, \ \forall x \in \mathbb{Z}_p[G]$$

However, cohomological conjugation endows $H^1(K, T)$ with a natural G-action. The way to see this action in $H^1(\mathbb{Q}, T \otimes \mathbb{Z}_p[G])$ is considering $T \otimes \mathbb{Z}_p[G]$ as a $\mathbb{Z}_p[G]$-module with the multiplication by the elements of G given by

$$\sigma(t \otimes x) = t \otimes x\sigma^{-1} \ \forall \sigma \in G, \ \forall t \in T, \ \forall x \in \mathbb{Z}_p[G]$$

When $T \otimes \mathbb{Z}_p[G]$ is a $\mathbb{Z}_p[G]$-module, then $H^1(\mathbb{Q}, T \otimes \mathbb{Z}_p[G])$ is also a $\mathbb{Z}_p[G]$-module. In this situation, Shapiro's lemma isomorphism respects the $\mathbb{Z}_p[G]$-structures.

Moreover, the Bloch-Kato local conditions behave well under the isomorphism from Shapiro's lemma, so we have an isomorphism

$$H^1_{\mathcal{F}_{\mathrm{cl}}}(K, T_pE) = H^1_{\mathcal{F}_{\mathrm{BK}}}(K, T)$$

The Galois structure of $H^1_{\mathcal{F}_{\mathrm{cl}}}(K, T_pE)$ is the structure of $H^1_{\mathcal{F}_{\mathrm{BK}}}(K, T)$ as a $\mathbb{Z}_p[G]$-module. However, when the order of G is not a power of p, $\mathbb{Z}_p[G]$ is not a local ring and we cannot apply the general theory of Kolyvagin systems developed in Chapter 3 directly.

We will only deal with the case when the order of G is prime to p. In that case, we can split the Selmer group into character parts. In order to do that, we tensor it with the ring of integers of the $\mathbb{Q}_p(\mu_d)$, which will be denoted by $\mathcal{O}_d$. After this process, we can split the Selmer group into character parts, obtaining an isomorphism.

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathbb{Z}_p[G]) \otimes \mathcal{O}_d = \bigoplus_{\chi \in \widehat{G}} H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes e_\chi \mathcal{O}_d[G])$$

where e_χ is the idempotent element associated with the character χ, defined in Definition 8.2.4 below, and $\widehat{G}$ denotes the set of characters of G.

When G is an abelian group, every character in $\widehat{G}$ is one-dimensional and $T \otimes e_\chi \mathcal{O}_d[G]$ is isomorphic, as a $\mathbb{Z}_p[G]$-module, to $T \otimes \mathcal{O}_d(\chi)$, where $\mathcal{O}_d(\chi)$ is $\mathcal{O}_d$ endowed with an action of G given by $\sigma x = \chi(\sigma)x$, we have an isomorphism

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathbb{Z}_p[G]) \otimes \mathcal{O}_d \cong \bigoplus_{\chi \in \widehat{G}} H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathcal{O}_d(\chi))$$

Therefore, we can study the groups $H^1(\mathbb{Q}, T \otimes \mathcal{O}_d(\chi))$ instead. The Galois structure of $H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathbb{Z}_p[G])$ is completely determined by the $\mathcal{O}_d$ structure of $H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathcal{O}_d(\chi))$ of every χ (see §8.5 for examples of this process).

Note that $\mathcal{O}_d$ is a discrete valuation ring and, when d is not divisible by p, then p is an uniformizer of the maximal ideal. Hence $\mathcal{O}_d/p^k$ satisfy Assumptions (CR1) for every $k \in \mathbb{Z}$, so the theory from Chapter 3 and, in particular, Theorem 3.3.7 can be applied in this case.

Therefore, we study the character parts individually, i.e., we fix a character χ of G and study the Galois representation

$$T(\chi) = T_pE \otimes \mathcal{O}_d(\chi) \tag{8.1}$$

We can see that $T(\chi)$ satisfies the assumption required to apply the theorems about Kolyvagin systems.

Proposition 8.1.4. Under Assumption (EC) and (Ab), $T(\chi)$ satisfies Assumption (BI).

Proof. It is easy to see that the hypotheses (T0) and (E0) coincide. Moreover, assumption (K1) in (Ab) implies that $\mathbb{Q}(T) \cap K = \mathbb{Q}$, since the class number of $\mathbb{Q}$ is one. Hence the image of the Galois representation $\rho_\chi : \ G_{\mathbb{Q}} \to \mathrm{Aut}(T(\chi))$ is

$$\mathrm{GL}_2(\mathbb{Z}_p) \otimes \mathcal{O}_d \subset \mathrm{GL}_2(\mathcal{O}_d)$$

Assumptions (T1) and (T2) follows from this. The image of ρ_χ also contains a subgroup Δ as in Remark 2.1.4, so the same argument proves (T3) in this case. □

We can also show that the Bloch-Kato Selmer structure on $T(\chi)$ has core rank zero, while the canonical structure has core rank one.

Proposition 8.1.5. The Bloch-Kato Selmer structure on $T(\chi)$ has core rank $\chi(\mathcal{F}_{\mathrm{BK}}) = 0$. In addition, the canonical Selmer structure has core rank $\chi(\mathcal{F}^{\mathrm{can}}) = 1$.

Proof. Note that $\Sigma_{\mathcal{F}_{\mathrm{BK}}} = \Sigma_{\mathcal{F}^{\mathrm{can}}} = \{p, \infty\}$. For the Bloch-Kato Selmer structure, we can compute using Proposition 8.1.3 that

$$\begin{array}{ll} \mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T(\chi))) = 1 & \mathrm{rank}_{\mathbb{Z}_p}(H^0(\mathbb{Q}_p, T(\chi))) = 0 \\ \mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{R}, T(\chi))) = 0 & \mathrm{rank}_{\mathbb{Z}_p}(H^0(\mathbb{R}, T(\chi))) = 1 \end{array}$$

Then Remark 5.3.6 implies that $\chi(\mathcal{F}_{\mathrm{BK}}) = 0$. For the canonical structure, we have that

$$\mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}^{\mathrm{can}}}(\mathbb{Q}_p, T(\chi))) = 2 \quad \mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{R}, T(\chi))) = 0$$

So Remark 5.3.6 implies, in this case, that $\chi(\mathcal{F}^{\mathrm{can}}) = 1$. □

The core ranks coincide when we consider the Iwasawa Selmer group.

Corollary 8.1.6. The core rank of $\mathcal{F}_{\mathrm{BK}}$ and $\mathcal{F}^{\mathrm{can}}$ on $T(\chi) \otimes_{\mathbb{Z}_p} \Lambda$ are 0 and 1.

Proof. Since both $\mathcal{F}_{\mathrm{BK}}$ and $\mathcal{F}^{\mathrm{can}}$ are cartesian on both $T(\chi)$ and $T(\chi) \otimes_{\mathbb{Z}_p} \Lambda$ are cartesian, the result follows from Proposition 8.1.5 and 6.1.17. □

Remark 8.1.7. By Proposition 5.3.8, the core ranks also remain unchanged after projecting the Selmer structures to the finite quotients $T(\chi)/p^k$.

Assuming we can compute the Fitting ideals of the Selmer group of every twisted Galois representation $T(\chi)$, the following result explains how to recover the Fitting ideals of the Selmer group over K as a $\mathbb{Z}_p[G]$-module.

Proposition 8.1.8. The Fitting ideals of $H^1_{\mathcal{F}_{\mathrm{BK}}}(K, T)$ can be computed as:

$$\mathrm{Fitt}^i_{\mathbb{Z}_p[G]}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T)\big) = \mathbb{Z}_p[G] \cap \sum_{\chi \in \widehat{G}} e_{\overline{\chi}} \,\mathrm{Fitt}^i_{\mathcal{O}_d}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathcal{O}_d(\chi))\big) \tag{8.2}$$

Moreover, there is an isomorphism

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T) \cong \bigoplus_i \mathbb{Z}_p[G]/I_i$$

where I_i are ideals of $\mathbb{Z}_p[G]$ satisfying that $I_{i-1} \subset I_i$ for all i and that

$$\mathrm{Fitt}^{i-1}_{\mathbb{Z}_p[G]} H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T) = I_i \,\mathrm{Fitt}^i_{Z_p[G]} H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T)$$

Proof. By the definition of the Fitting ideals, we can compute

$$\mathrm{Fitt}^i_{\mathbb{Z}_p[G]}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T)\big) = \mathrm{Fitt}^i_{\mathcal{O}_d[G]}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T)\otimes\mathcal{O}_d\big)$$

Since $\mathcal{O}_d[G] = \bigoplus_{\chi\in\widehat{G}} e_\chi\mathcal{O}_d[G]$, then

$$\mathrm{Fitt}^i_{\mathcal{O}_d[G]}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K,T)\otimes\mathcal{O}_d\big) = \sum_{\chi\in\widehat{G}} e_{\overline{\chi}}\ \mathrm{Fitt}^i_{\mathcal{O}_d}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(Q,T\otimes\mathcal{O}_d(\chi))\big)$$

Then (8.2) follows from [AM69, Proposition 1.17] since every ideal of $\mathbb{Z}_p[G]$ is a contracted ideal under the inclusion $\mathbb{Z}_p[G]\hookrightarrow\mathcal{O}_d[G]$. Indeed, if $G = C_{n_1}\times\cdots\times C_{n_s}$, where C_j represents the cyclic group of j elements and $n_i \mid n_{i-1}$ for all i. Hence,

$$\mathbb{Z}_p[G]\cong\mathbb{Z}_p[C_{n_1}][C_{n_2}]\cdots[C_{n_s}]$$

If $\mathcal{O}$ is an unramified discrete valuation ring with with residue field k and residue characteristic p and j is prime to p, then

$$\mathcal{O}[C_j]\cong\mathcal{O}[T]/(T^j-1)\cong\bigoplus_k\mathcal{O}[T]/(f_k(T))$$

where $f_k(T)$ are the irreducible factors of T^j-1, which are all distinct. By Gauss's and Hensel's lemmas, the reductions $\overline{f_k}(T)$ are irreducible in $k[T]$ and, therefore, $\mathcal{O}[T]/(f_k(T))$ are unramified discrete valuation rings.

By repeating this process for every cyclic factor of G, we obtain a decomposition

$$\mathbb{Z}_p[G]\cong\bigoplus_k\mathcal{O}_k$$

where every $\mathcal{O}_{k'}$ is an unramified discrete valuation ring, associated with a Galois orbit the characters $G\to\overline{\mathbb{Z}_p}$.

The above decomposition implies that every non-zero ideal in $\mathbb{Z}_p[G]$ is the product of maximal ideals. Since the zero ideal is contracted because the map $\mathbb{Z}_p[G]\to\mathcal{O}_d[G]$ is injective, we just need to prove that the maximal ideals of $\mathbb{Z}_p[G]$ are contracted. Every maximal ideal of $\mathbb{Z}_p[G]$ is of the form

$$\mathfrak{m}_i = \bigoplus_{k\neq i}\mathcal{O}_k\oplus p\mathcal{O}_i$$

If e_i is the idempotent element associated with $\mathcal{O}_i$, then there are some characters $\chi_{i_1}, \ldots, \chi_{i_t} \in \widehat{G}$ such that

$$e_i = \sum_{j=1}^{t} e_{\chi_{i_1}}$$

Hence

$$m_i = \mathbb{Z}_p[G] \cap \left[\left(\sum_{j=1}^{t} e_{\chi_{i_1}} \right) p\mathcal{O}_d[G] + \left(1 - \sum_{j=1}^{t} e_{\chi_{i_1}} \right) \mathcal{O}_d[G] \right]$$

The second part of this proposition follows by applying the structure theorem of finitely generated modules over principal ideal domains to the discrete valuation rings in the decomposition of $\mathbb{Z}_p[G]$. $\square$

8.1.3 Modular symbols

By the modularity theorem, there exist morphisms

$$\varphi_0 : \; X_0(N) \to E, \quad \varphi_1 : \; X_1(N) \to E$$

where $X_0(N)$ and $X_1(N)$ are the modular curves associated with the groups $\Gamma_0(N)$ and $\Gamma_1(N)$, respectively. We may choose φ_0 and φ_1 of minimal degree.

The modularity theorem also proves the existence of a newform f of weight 2 and level N associated with the isogeny class of E. Fixing a Néron differential ω_E, there exist some constants $c_0(E)$ and $c_1(E)$ such that[1]

$$\varphi_0^*(\omega) = c_0(E) 2\pi i f(\tau)\, d\tau, \quad \varphi_1^*(\omega) = c_1(E) 2\pi i f(\tau)\, d\tau$$

The Manin constant $c_0(E)$ is an integer (see [Edi91, Proposition 2]) and is conjecturally equal to 1 if E is an X_0-optimal elliptic curve, i.e., the degree of φ_0 is minimal among all modular parametrizations of curves in the isogeny class of E (see [Man72, §5]). The conjecture was proven in [Maz78] if E is semistable. In particular, it is proven that, for an X_0-optimal elliptic curve, $c_0(E)$ is only divisible by 2 and primes of additive reduction. Note that (E1) implies E does not admit any p-isogeny, so $c_0(E)$ is conjecturally prime to p.

The constant $c_1(E)$ was conjectured to be 1 for all elliptic curves by G. Stevens in [Ste89, conjecture 1].

By the modularity theorem, we can define for an elliptic curve the modular symbols of its associated modular form.

[1] We may choose φ_0 and φ_1 in a way that both $c_0(E)$ and $c_1(E)$ are positive.

Notation 8.1.9. We denote by $f = f_E = \sum_{n=1}^{\infty} a_n q^n$ the newform associated with the isogeny class of E by the modularity theorem. Note that, for every prime number p not dividing the conductor N, the Fourier coefficient a_p is given by the formula

$$a_p = p + 1 - \#\widetilde{E}(\mathbb{F}_p)$$

Definition 8.1.10. The modular symbol of E for some $\frac{a}{m} \in \mathbb{Q}$ is defined as

$$\lambda\left(\frac{a}{m}\right) := \int_{i\infty}^{\frac{a}{m}} 2\pi i f_E(z)\, dz$$

where the integral follows the vertical line in the upper half plane from the cusp at infinity to $\frac{a}{m} \in \mathbb{Q}$.

Proposition 8.1.11. ([WW22, lemma 5]) For every $\frac{a}{m} \in \mathbb{Q}$, the modular symbols satisfy the following relation:

$$\lambda\left(\frac{-a}{m}\right) = \overline{\lambda\left(\frac{a}{m}\right)}$$

We can consider the real and imaginary parts of the modular symbols and normalise them by one of the Néron periods $\Omega_E^{\pm}$ to obtain rational numbers carrying important arithmetic information.

Definition 8.1.12. The signed algebraic modular symbols are the normalised real and imaginary parts of the modular symbols:

$$\left[\frac{a}{m}\right]^{\pm} := \frac{\lambda\left(\frac{a}{m}\right) \pm \lambda\left(\frac{-a}{m}\right)}{2\Omega_E^{\pm}} \in \mathbb{Q}$$

where $\Omega_E^{\pm}$ denotes the Néron period.

Remark 8.1.13. Assumption (E2) is required to ensure that the denominator of $\left[\frac{a}{m}\right]^{\pm}$ is not divisible by p (see [Ste89, Theorem 4.6]) Thus

$$\left[\frac{a}{m}\right]^{\pm} \in \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$$

Modular symbols are closely related to the twisted L-values of the elliptic curve, which are defined as the motivic L-values of the twisted Galois representation, as defined in Definition A.1.6.

Definition 8.1.14. The χ-twisted L-function $L(E, \chi)$ is defined as the motivic L-function (see Definition A.1.6) of the twisted Galois representation $T(\chi)$. For $\mathrm{Re}(s) \geq 3/2$, it can be computed as the infinite product

$$L(E, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n}{n^s} = \prod_{\ell} \left(1 - \ell^{-s} a_\ell + \mathbf{1}_N(\ell)\ell^{1-2s}\right)^{-1}$$

Proposition 8.1.15. ([MTT86, p. I.8.6]) For every character χ of conductor c, the twisted special L-value normalised by a Néron period can be expressed as a linear combination of modular symbols:

$$\frac{L(E, \chi, s)}{\tau(\overline{\chi})\Omega_E^{\chi(-1)}} = \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \overline{\chi}(a) \left[\frac{a}{c}\right]^{\chi(-1)}$$

where $\tau(\overline{\chi})$ is the Gauss sum of the conjugate character $\overline{\chi}$.

8.1.4 Iwasawa main conjecture

We will now make comments on Assumption (K5). It refers to the Iwasawa main conjecture of a twisted modular form. We start by defining the twisting process of modular forms.

Definition 8.1.16. Let $f \in S_2(Y_0(N))$ and let χ be a Dirichler character of conductor c. We define the twist

$$f_\chi = \sum_{n=1}^{\infty} \chi(n) a_n q^n \in S_2(Y_1(Nc^2), \chi^2)$$

To the modular form f_χ, there is an associated Galois representation V_{f_χ}, which is an $\mathcal{O}_d \otimes \mathbb{Q}_p$-vector space in which, for every prime ℓ not dividing Nc, the Frobenius elements Frob_ℓ act with characteristical polynomial

$$P_\ell = X^2 - a_\ell \chi(\ell) X + \ell \chi^2(\ell)$$

By Brauer-Nesbitt theorem, V_{f_χ} is isomorphic, as a Galois representation, to $T(\chi) \otimes \mathbb{Q}_p$. The irreducibility assumptions on T_pE, deduced from (E1), follows to the uniqueness of Galois equivariant lattices in V_{f_χ} up to scalar multiplication.

The L-function of f_χ is defined as the L-function of its Galois representation, constructed in Definition A.1.6, so it coincides with the twisted L-function $L(E, \chi, s)$. It can be defined by the infinite product

$$L(f_\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n}{n^s} = \prod_{\ell} \left(1 - \ell^{-s}\chi(\ell) a_\ell + \mathbf{1}_{Nc}(\ell)\ell^{1-2s}\chi(\ell)^2\right)^{-1}$$

where $\mathbf{1}_{Nc}$ is the trivial Dirichlet character modulo Nc, with takes the value 1 at numbers that are coprime with Nc and 0 otherwise.

Remark 8.1.17. We know comment on Assumption (K5) about the main conjecture for the modular form f_χ. For its Galois representation, there is a canonical Euler system obtained as the twist of the Euler system in the Tate module of the elliptic curve associated with the Néron periods, which was constructed in Theorem A.3.2. The interested reader will find in Remark 8.2.20 how this Euler system can be obtained from the more general Kato's construction described in Theorem A.3.5. Assumption (K5) refers to the Iwasawa main conjecture of this Euler system, as stated in Definition A.2.3.

Remark 8.1.18. There is an equivalent formulation of the Iwasawa main conjecture that is independent of the Néron period (see [Kat04, Conjecture 12.10]).

Remark 8.1.19. There is an equivalent, more commonly known, formulation of the Iwasawa main conjecture in terms of the p-adic L-function associated with a modular form, first constructed in [MS74].

Although Assumption (K5) only concerns the Iwasawa main conjecture localised at the ideal (X), the full Iwasawa main conjecture is known under mild technical hypothesis, as detailed below.

Remark 8.1.20. Iwasawa main conjecture for f_χ was proven in [SU14, Theorem 1] when p does not divide the level of f_χ, which, under our assumption (K1), is equivalent to E having good reduction at p, and the existence of an auxilliary prime $\ell \mid\mid N$ such that χ is unramified at ℓ and the reduction modulo ℓ of the Galois representation $\overline{\rho}_{f_\chi}$ satisfies that $\dim_{\mathbb{F}_\ell} \overline{\rho}_{f_\chi}^{\mathcal{I}_\ell} = 1$ and $\dim_{\mathbb{F}_\ell} \overline{\rho}_{f_\chi}^{G_{\mathbb{Q}_\ell}} = 0$.

The proof of the Iwasawa main conjecture was extended in [FW22, Theorem 1.1] to some cases in which E has bad reduction at p, assuming the existence of the above auxiliary prime ℓ and that the semisimplification of $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is different to $\psi \oplus \psi$ and $\psi \oplus \chi_{\mathrm{cyc}}\psi$ for any character ψ and the cyclotomic character χ_{cyc}.

8.1.5 Structure of the Selmer group

From the comments above, we will study the Galois representation

$$T(\chi) := T_pE \otimes \mathcal{O}_d(\chi)$$

whose coefficient ring is the discrete valuation ring $\mathcal{O}_d$. We will be interested in computing the Bloch-Kato Selmer group (see Definition 2.3.4) associated with this representation, although other Selmer structures like the canonical one, or the one with the restricted local conditions at p will play important roles here.

For this representation, there is an Euler system constructed by twisting Kato's Euler system (see Definition 8.2.18). The Kolyvagin derivative (see Corollary A.4.11) produces a Kolyvagin system in $\overline{\mathrm{KS}}(T(\chi), \mathcal{F}^{\mathrm{can}})$, which can be seen as an ultra Kolyvagin system $\kappa \in \mathbf{KS}(T(\chi), \mathcal{F}^{\mathrm{can}})$ by Theorem 5.2.19. It will be used to apply Theorems 5.4.5 and 5.4.6 to compute the Fitting ideals of the Selmer group.

In order to apply those results, whenever $n \in \mathcal{N}(T(\chi)/p^k)$, we give a description of the quantities

$$\mathrm{ind}(\mathrm{loc}_p(\kappa_n), H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T(\chi)/p^k))$$

in terms of the modular symbols. For this representation, we can give an explicit description of the Kolyvagin primes for a suitable choice of τ.

Remark 8.1.21. For every $k \in \mathbb{N}$, the set of primes $\mathcal{P}(T(\chi)/p^k)$ is the set of primes ℓ satisfying the following conditions.

- (PE0) E has good reduction at ℓ and ℓ is unramified in $K/\mathbb{Q}$.
- (PE1) $\ell \equiv 1 \mod p^k$.
- (PE2) $\widetilde{E}_\ell(\mathbb{F}_\ell)[p^k]$ is free of rank one over $\mathbb{Z}/p^k$.
- (PE3) ℓ splits completely in $K/\mathbb{Q}$.

The indices of the localisation of the Kolyvagin classes can be computed using a twisted version of the Kurihara numbers. Kurihara numbers were defined by M. Kurihara in [Kur14a] and [Kur14b]. In those articles, they were related to the structure of the Selmer group $\mathrm{Sel}(\mathbb{Q}, E[p^\infty])$ and the veracity of the Iwasawa main conjecture. The definition given by M. Kurihara was the particular case of our definition 8.1.22 below when χ is the trivial character. In this definition, we twist the Kurihara numbers by the different characters χ of G, so we can get information about the structure of $H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T \otimes \mathcal{O}_d(\chi))$.

Definition 8.1.22. Let $n \in \mathcal{N}(T(\chi)/p^k)$ and let χ be a Dirichlet character of conductor c. The *twisted Kurihara number* is defined as

$$\widetilde{\delta}_{n,\chi} := \sum_{a\in(\mathbb{Z}/cn\mathbb{Z})^\times} \overline{\chi}(a)\left[\frac{a}{cn}\right]^{\chi(-1)}\left(\prod_{\ell|n}\log^p_{\eta_\ell}(a)\right) \in \mathcal{O}_d/p^{k_n} \qquad (8.3)$$

where η_ℓ is a generator of $(\mathbb{Z}/\ell)^\times$ and $\log^p_{\eta_\ell}(a)$ is the unique $x \in \mathbb{Z}/p^k$ such that $\eta_\ell^{-x}a$ has order prime to p in $(\mathbb{Z}/\ell)^\times$ (see definition 8.3.10). Here k_n is the minimal $k \in \mathbb{N}$ such that $n \in \mathcal{N}_k$.

Remark 8.1.23. The definition of the twisted Kurihara numbers depends on the choice of the primitive roots η_ℓ for the prime divisors ℓ of n. However, the p-adic valuation of $\delta_{n,\chi}$ is independent of this choice.

Remark 8.1.24. Note that for $n = 1$, the Birch formula in [MTT86, (I.8.6)] relates the twisted Kurihara number with the twisted special L-value:

$$\widetilde{\delta}_{1,\chi} = \sum_{a\in(\mathbb{Z}/c\mathbb{Z})^*} \overline{\chi}(a)\left[\frac{a}{c}\right]^{\chi(-1)} = \frac{1}{\tau(\overline{\chi})}\frac{L(E,\chi,1)}{\Omega_E^{\chi(-1)}}$$

where $\tau(\chi)$ is the Gauss sum of χ.

Twisted Kurihara numbers generate the theta ideals associated with Kato's Kolyvagin system.

Theorem 8.1.25. Let $\kappa \in \mathrm{KS}(T(\chi)/p^k, \mathcal{F}^{\mathrm{can}})$ be the Kolyvagin system obtained from the twisted Kato's Euler system (see Remark 8.2.20). Then $\Theta_i^{(0)}(\kappa, \mathcal{F}_{\mathrm{BK}})$ is the ideal generated for all the Kurihara numbers $\delta_{n,\chi}$ of the vertices $n \in \mathcal{N}^i(T(\chi)/p^k)$.

Proof. It follows the following equality, proven in Theorem 8.3.17 below:

$$(\widetilde{\delta}_{n,\chi})\mathcal{O}_d(\chi)/p^k = \mathrm{ind}(\mathrm{loc}_p(\kappa_n), H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T(\chi)/p^k)) \qquad \square$$

We can define analytic theta ideals associated with the collection of Kurihara numbers.

Definition 8.1.26. Fix $k \in \mathbb{N}$. For every $i \in \mathbb{Z}_{\geq 0}$, define the k^{th} *analytic theta ideal* as ideal in $\mathcal{O}_d(\chi)/p^k$ generated by the twisted Kurihara numbers:

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) := \sum_{n\in\mathcal{N}_i(T(\chi)/p^k)} \big(\widetilde{\delta}_{n,\chi}\big)$$

In addition, the analytic theta ideal at infinity is defined as

$$\Theta_\infty\big(\widetilde{\delta}_\chi^{(k)}\big) := \sum_{n\in\mathcal{N}(T(\chi)/p^k)} \big(\widetilde{\delta}_{n,\chi}\big)$$

The central result in this section relates the analytic theta ideals with the ones associated with Kato's Kolyvagin system.

Theorem 8.1.27. For all $i \in \mathbb{Z}_{\geq 0}$, we have that

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \Theta_i^{(0)}(\kappa_{\mathrm{Kato}}^{(k)}, \mathcal{F}_{\mathrm{BK}})$$

The proof of this theorem will be given in §8.3.3 (see Remark 8.3.18).

We can now apply Theorems 3.3.7 and 3.4.1 to compute the structure of the Selmer group. Note that the following theorem has a slightly different conclusion than Theorems 3.3.7, being the reason behind functional equations on the analytic theta ideals. The following theorem is proven in §8.4.

Theorem 8.1.28. Let $E/\mathbb{Q}$ and $p \geq 5$ be an elliptic curve and a prime number satisfying (E1)-(E2) and let $K/\mathbb{Q}$ be an abelian extension satisfying (K1)-(K5). Then, for a large enough $k \in \mathbb{N}$, the ideal $\Theta_\infty\big(\widetilde{\delta}_\chi^{(k)}\big)$ is a non-zero ideal. In addition, we have the following:

- If $\chi \neq \overline{\chi}$, then

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \mathrm{Fitt}^i_{\mathcal{O}_d(\chi)/p^k}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, E[p^k][\chi])\big)\Theta_\infty\big(\widetilde{\delta}_\chi^{(k)}\big)$$

- If $\chi = \overline{\chi}$ and i has a different parity as the quantity $r\chi(-N)$, where r is the rank of the Mordell-Weil group $E(\mathbb{Q})$, then $\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = 0$.
- If $\chi = \overline{\chi}$ and i has the same parity as $r\chi(-N)$, then

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \mathrm{Fitt}^i_{\mathcal{O}_d(\chi)/p^k}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, E[p^k][\chi])\big)\Theta_\infty\big(\widetilde{\delta}_\chi^{(k)}\big)$$

We can generalise this result to compute the Fitting ideals of the Selmer group of $E[p^\infty]$. We start by defining patched Kurihara numbers.

Definition 8.1.29. Let $\mathfrak{n} \in \mathcal{N}(T(\chi))$ be represented by the sequence $(n_i)_{i\in\mathbb{N}}$. The *patched Kurihara number* $\delta_{\mathfrak{n},\chi}$ is defined as the patching of $\delta_{n_i,\chi}$. More precisely, for every $k \in \mathbb{N}$, $n_i \in \mathcal{N}(T(\chi)/p^k)$ for $\mathcal{U}$-many i, so $\delta_{n_i,\chi}$ is a well-defined element in $\mathcal{O}_d(\chi)/p^k$. By Lemma 4.1.21, we can consider the element

$$\mathcal{U}_i(\delta_{n_i,\chi}) \in \mathcal{O}_d(\chi)/p^k$$

This construction is clearly compatible under the projection maps, so the limit will be a well defined quantity

$$\delta_{\mathfrak{n},\chi} \in \varprojlim_{k\in\mathbb{N}} \mathcal{O}_d(\chi)/p^k = \mathcal{O}_d(\chi)$$

We can generalise the definition of the analytic theta ideals.

Definition 8.1.30. For every $i \in \mathbb{Z}_{\geq 0}$, define the *patched analytic theta ideal* as the ideal of $\mathcal{O}_d$ defined by

$$\mathbf{\Theta}_i\big(\widetilde{\delta}_\chi\big) := \sum_{\mathfrak{n}\in\mathcal{N}_i(T(\chi))} \big(\widetilde{\delta}_{\mathfrak{n},\chi}\big)$$

In addition, the analytic theta ideal at infinity is defined as

$$\mathbf{\Theta}_\infty\big(\widetilde{\delta}_\chi\big) := \sum_{n\in\mathcal{N}(T(\chi))} \big(\widetilde{\delta}_{\mathfrak{n},\chi}\big)$$

Remark 8.1.31. The patched analytic theta ideals can be constructed as the limit of classical analytic theta ideals, in a similar way as Theorem 5.2.30.

The analogue of Theorem 8.1.28 holds for the patched theta ideals. We will not write it explicitly, but will instead show the following, equivalent description of the Selmer group up to isomorphism.

Theorem 8.1.32. Let $E/\mathbb{Q}$ and $p \geq 5$ be an elliptic curve and a prime number satisfying (E1)-(E2) and let $K/\mathbb{Q}$ be an abelian extension satisfying (K1)-(K5). Then $\mathbf{\Theta}_\infty\big(\widetilde{\delta}_\chi\big)$ is non-zero.

Call r to the minimum i such that $\mathbf{\Theta}_i(\widetilde{\delta}_\chi) \neq 0$ and s to the minimum j such that $\mathbf{\Theta}_j(\widetilde{\delta}_\chi) = \mathcal{O}_d(\chi)$.

In addition, denote by n_i the exponent satisfying that

$$\mathbf{\Theta}_i(\widetilde{\delta}_\chi) = (p)^{n_i}\mathbf{\Theta}_\infty(\widetilde{\delta}_\chi)$$

Furthermore, denote $a_i := \frac{n_i - n_{i+2}}{2}$ and $b_i := n_i - n_{i+1}$.

- If $\chi = \overline{\chi}$, then $s - r \in 2\mathbb{Z}$ and

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, E[p^\infty](\chi)\otimes\mathcal{O}_d)^\vee \cong \mathcal{O}_d^r \oplus \bigoplus_{i=1}^{\frac{s-r}{2}} \left(\frac{\mathcal{O}_d}{(p)^{a_{r+i-2}}}\right)^2$$

- If $\chi \neq \overline{\chi}$, then

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T\otimes\mathcal{O}_d(\chi)) \cong \mathcal{O}_d^r \oplus \bigoplus_{i=1}^{s-r} \frac{\mathcal{O}_d}{p^{b_{r+i-1}}}$$

Remark 8.1.33. The particular case of Theorem 8.1.32 in which $K = \mathbb{Q}$ is a result of C-H. Kim in [Kim25].

Remark 8.1.34. It is enough to prove Theorem 8.1.32 for the primitive characters of G, i.e., those of conductor c. Indeed, if χ is a character of conductor m, consider $K_m = K \cap \mathbb{Q}(\mu_m)$. Note that χ is a primitive character of $\mathrm{Gal}(K_m/\mathbb{Q})$. Since $[K : K_m]$ is prime to p,

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(K_m, T(\chi)) \cong H^1_{\mathcal{F}_{\mathrm{BK}}}(K, T(\chi))^{\mathrm{Gal}(K/K_m)}$$

Hence the χ parts of both Selmer group coincide. Hence if Theorem 8.1.32 holds for K_m and χ, it also holds for K and χ.

8.1.6 Kurihara conjecture

When χ is the trivial character, M. Kurihara conjectured in [Kur14b] the existence of some $n \in \mathcal{N}$ such that $\mathrm{ord}(\delta_n)$ is equal to zero. When p is a prime of ordinary reduction, M. Kurihara proved it under the assumptions of the non-degeneracy of the p-adic height pairing and the Iwasawa main conjecture (see [Kur14a, theorem B]). An analogue statement can be conjectured for other characters.

Kurihara's conjecture was proven to be equivalent to the Iwasawa main conjecture by R. Sakamoto proved in [Sak22, Theorem 1.2] (see also [Kim25, Theorem 1.11]).

Here, we state a twisted version of Kurihara conjecture, which can serve as a numerical check of the Iwasawa main conjecture for the twisted modular form f_χ.

Conjecture 8.1.35. (Twisted Kurihara conjecture) There exists some $n \in \mathcal{N}(T(\chi)/p)$ such that $(\delta_{n,\chi}) \notin p\mathcal{O}_d(\chi)$.

Remark 8.1.36. Note that Conjecture 8.1.35 is equivalent to its analogue considering patched Kurihara numbers. It is also equivalent to

$$\boldsymbol{\Theta}_\infty(\widetilde{\delta_\chi}) = \mathcal{O}_d$$

The twisted Kurihara conjecture can be proven to be equivalent to an Iwasawa main conjecture.

Theorem 8.1.37. Let $E/\mathbb{Q}$ and $p \geq 5$ be an elliptic curve and a prime number satisfying (E1)-(E2), let $K/\mathbb{Q}$ be an abelian extension satisfying (K1)-(K5) and let χ be primitive a character of $\mathrm{Gal}(K/\mathbb{Q})$. Then Kurihara conjecture 8.1.35 and the Iwasawa main conjecture for the twisted Kato's Euler system (see Definition A.2.3 and Remark 8.2.20).

Theorem 8.1.37 will be proven in §8.3.4.

8.2 The Euler system

8.2.1 Dual exponential map

K. Kato constructed in [Kat04] an Euler system for the Tate module of an elliptic curve $T = T_pE$ and defined an exponential map to relate this Euler system to the special values of the twisted L-functions of the elliptic curve.

Using the Néron differential ω_E we have already fixed, we can define the dual exponential map (see [BK90, definition 3.10])

$$\exp^*_{\omega_E} : \ H^1_{/\mathcal{F}_{\mathrm{BK}}}(F_\mathfrak{p}, T) \otimes \mathbb{Q}_p \to F_\mathfrak{p}$$

where $F_\mathfrak{p}$ is the completion at a prime $\mathfrak{p}$ above p of an abelian extension $F/\mathbb{Q}$ and $H^1_{/\mathcal{F}_{\mathrm{BK}}}(F_\mathfrak{p}, T)$ is the quotient $H^1(F_\mathfrak{p}, T)/H^1_{\mathcal{F}_{\mathrm{BK}}}(F_\mathfrak{p}, T)$.

We will sketch the construction of the exponential map from [BK90]. The following is the fundamental short exact sequence from p-adic Hodge theory:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{\mathrm{crys}}^{\varphi=1} \oplus B_{\mathrm{dR}}^+ \longrightarrow B_{\mathrm{dR}} \longrightarrow 0 \tag{8.4}$$

where B_{crys} and B_{dR} are the crystalline and de Rham period rings.

Notation 8.2.1. For $V := T \otimes \mathbb{Q}_p$, we denote

$$\begin{aligned} D_{\mathrm{crys}}(V) &:= (B_{\mathrm{crys}} \otimes V)^{G_{F_\mathfrak{p}}} \\ D_{\mathrm{dR}}(V) &:= (B_{\mathrm{dR}} \otimes V)^{G_{F_\mathfrak{p}}} \\ D_{\mathrm{dR}}(V)^+ &:= (B_{\mathrm{dR}}^+ \otimes V)^{G_{F_\mathfrak{p}}} \end{aligned}$$

If we consider the cohomological exact sequence in (8.4) tensored with V,

$$\mathbb{Q}_p \rightarrowtail D_{\mathrm{crys}}(V)^{\varphi=1} \oplus D_{\mathrm{dR}}(V)^+ \longrightarrow D_{\mathrm{dR}}(V) \twoheadrightarrow H^1_{\mathcal{F}_{\mathrm{BK}}}(F_\mathfrak{p}, V)$$

Since the tangent space of E is isomorphic to $D_{\mathrm{dR}}(V)/D_{\mathrm{dR}}(V)^+$, this exact sequence induces a surjective map

$$\exp: \ \tan(E/F_{\mathfrak{p}}) \to H^1_{\mathcal{F}_{\mathrm{BK}}}(F_{\mathfrak{p}}, V)$$

The dual ω_E^* of the Néron differential generates the tangent space of $E/F_{\mathfrak{p}}$ as an $F_{\mathfrak{p}}$-vector space, so we can consider the exponential as a map from $F_{\mathfrak{p}}$,

$$\exp_{\omega_E}: \ F_{\mathfrak{p}} \to H^1_{\mathcal{F}_{\mathrm{BK}}}(F_{\mathfrak{p}}, V)$$

Its dual map is the one we will be interested in.

$$\exp^*_{\omega_E}: \ H^1_{/\mathcal{F}_{\mathrm{BK}}}(F_{\mathfrak{p}}, V) \to \mathrm{Hom}(F_{\mathfrak{p}}, \mathbb{Q}_p)$$

We can identify the latter with $F_{\mathfrak{p}}$ via the following isomorphism

$$F_{\mathfrak{p}} \to \mathrm{Hom}(F_{\mathfrak{p}}, \mathbb{Q}_p), \ x \mapsto (y \mapsto \mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(xy)) \tag{8.5}$$

Remark 8.2.2. The above map is defined for every Galois representation using p-adic Hodge theory. However, when T is the Tate module of an elliptic curve E, the exponential map has a geometrical meaning (see [BK90, example 3.10.1]). Note that in this case

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(F_{\mathfrak{p}}, T) = \varprojlim_n E(F_{\mathfrak{p}})/p^n$$

Then the exponential map coincides with the Lie group exponential map defined on the elliptic curve. Moreover, it can be also understood as the tensor with $\mathbb{Q}_p$ of the formal group exponential map defined on certain power of the maximal ideal of the ring of integers of $F_{\mathfrak{p}}$.

We will be interested in the image under this map of elements coming from the global cohomology group $H^1(F, T)$. In order to do that, we consider the localisation at a rational prime q as the direct sum of the localisation maps above every prime v above q

$$\mathrm{loc}^s_q: \ H^1(F, V) \to \bigoplus_{v|q} H^1_{/\mathcal{F}_{\mathrm{BK}}}(F_v, V)$$

Then there is an Euler system $(z_F)_{F\in\Omega}$, which was constructed in Theorem A.3.2, satisfying the following interpolation property. Fix an inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$; for every character ψ of $\mathrm{Gal}(F/\mathbb{Q})$, we have that

$$\sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})} \psi(\sigma)\sigma(\exp^*_{\omega_E}(\mathrm{loc}^p_s(z_F))) = \frac{L_{S_F\cup\{p\}}(E, \psi, 1)}{\Omega^{\psi(-1)}} \in F \otimes \mathbb{Q}_p \tag{8.6}$$

where S_F is the primes ramifying at $F/\mathbb{Q}$ and $\Omega^\pm$ denote the Néron periods of E. Here $L_{S_F\cup\{p\}}(E,\psi,1)$ is the $S_F\cup\{p\}$-truncated and ψ-twisted L-function, defined as

$$L_{S_F\cup\{p\}}(E,\psi,s)=\prod_{\ell\notin S_F,\ell\neq p}(1-a_\ell\psi(\ell)\ell^{-s}+\mathbf{1}_N(\ell)\psi(\ell)^2\ell^{1-2s})^{-1}$$

8.2.2 Dual exponential of Kato's Euler system

The section is dedicated to give a computation of the dual exponential map of Kato's Euler system, building from the interpolation property in Theorem A.3.2. In this direction, we will show that it is the result of an action of a Stickelberger element in the group algebra acting on certain cyclotomic units.

Notation 8.2.3. Denote

$$w_F:=\exp^*_{\omega_E}(\mathrm{loc}_p^s(z_F))$$

We want to encode all the twisted L-values in an element of the group algebra. In order to do that, we define the idempotent elements associated with Dirichlet characters

Definition 8.2.4. Let ψ be a character of $\mathrm{Gal}(F/\mathbb{Q})$. Its *idempotent element* is defined as

$$e_\psi:=\frac{1}{[F:\mathbb{Q}]}\sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\overline{\psi}(\sigma)\sigma\in\mathbb{Z}_p[\psi][\mathrm{Gal}(F/\mathbb{Q})]$$

where $\mathbb{Z}_p[\psi]$ is the ring obtained by adjoining the values of ψ to $\mathbb{Z}_p$.

Remark 8.2.5. The interpolation property in (8.6) can be rewritten in terms of the idempotent elements in the following form. When ψ is a primitive character of $F/\mathbb{Q}$, the following holds.

$$e_\psi w_F=\frac{1}{[F:\mathbb{Q}]}\frac{L_{S_F\cup\{p\}}(E,\overline{\psi},1)}{\Omega^{\psi(-1)}}\tag{8.7}$$

Notation 8.2.6. For every $n,m\in\mathbb{N}$ such that $m\mid n$, we use the following notation. We call $\tilde{n}$ the product of all the primes dividing n and $r(m,n):=\mathrm{lcm}(\tilde{n},m)$. Furthermore, let $s(m,n):=\frac{r(m,n)}{m}$ be the product of primes dividing n but not m. Note that m and $s(m,n)$ are relatively prime. When there is no risk of confusion, we will denote these quantities by r and s.

Notation 8.2.7. Fix an embedding $\overline{\mathbb{Q}} \cong \mathbb{C}$ and denote by

$$\zeta_n := e^{\frac{2\pi i}{n}}$$

Taking this notation into account, we can now define the algebraic L-value.

Definition 8.2.8. Let $n \in \mathbb{N}$ be divisible by neither p nor any bad prime of E and ψ be a Dirichlet character conductor $m \mid n$. The *algebraic L-value* is defined as

$$\mathcal{L}_n(E, \psi) := \frac{L_{S_n \cup \{p\}}(E, \psi, 1)}{\varphi(n) e_{\overline{\psi}}(\zeta_r) \Omega^{\psi(-1)}} = \frac{(-1)^{\nu(s)} \overline{\psi}(s) \varphi(r) L_{S_n \cup \{p\}}(E, \psi, 1)}{\varphi(n) \varphi(m) e_{\overline{\psi}}(\zeta_m) \Omega^{\psi(-1)}}$$

where S_n is the set of prime divisors of n, $r = r(m, n)$, and φ represents the Euler totient function and $\zeta_j = e^{\frac{2\pi i}{j}} \in \mathbb{C}$.

Remark 8.2.9. The above is a modification of the definition in [WW22] to consider the cases when ψ is a non-primitive character, i.e., when $m < n$. Note that, when ψ is primitive, the product $\varphi(n) e_{\overline{\psi}}(\zeta_r)$ coincides with the Gauss sum of ψ, denoted by $\tau(\psi)$. In this case, we have that $m = n = r$, so $s = 1$. The algebraic L-value can be then computed as

$$\mathcal{L}_n(E, \psi) := \frac{L_{S_n \cup \{p\}}(E, \psi, 1)}{\tau(\psi) \Omega^{\psi(-1)} \in \overline{\mathbb{Q}}}$$

Remark 8.2.10. It is worth mentioning that the last equality comes from the trace-compatibility relations of the cyclotomic units. Note that

$$e_{\overline{\psi}}(\zeta_r) = \frac{\varphi(m)}{\varphi(r)} e_{\overline{\psi}} \Big(\mathrm{Tr}_{\mathbb{Q}(\mu_r)/\mathbb{Q}(\mu_m)}(\zeta_r) \Big) = \frac{\varphi(m)}{\varphi(r)} e_{\overline{\psi}} \Big((-1)^{\nu(s)} \zeta_m^{(s^{-1})} \Big)$$

Where s^{-1} is the inverse of s mod m. We then have that

$$e_{\overline{\psi}}(\zeta_r) = \frac{\varphi(m)}{\varphi(r)} e_{\overline{\psi}} \Big(\mathrm{Tr}_{\mathbb{Q}(\mu_r)/\mathbb{Q}(\mu_m)}(\zeta_r) \Big) = \frac{(-1)^{\nu(s)} \varphi(m)}{\overline{\psi}(s) \varphi(r)} e_{\overline{\psi}}(\zeta_m)$$

By the construction of the Néron periods, the algebraic L-values are elements in $\mathbb{Q}_p[\chi]$. However, we can use the results in [WW22] to prove integrality properties of the Néron periods, by bounding below the p-adic valuation of the algebraic L-values.

Notation 8.2.11. Denote

$$k_\psi := v_p \left(\frac{1 - a_p \psi(p) + \mathbf{1}_N(p) \psi(p)^2}{p} \right)$$

Proposition 8.2.12. Let $m \in \mathbb{N}$ be an integer prime to Np and let ψ be a character of conductor m, satisfying that $r(m,n) = n$ or, equivalently, that $\gcd(m, n/m) = 1$. Then

$$\psi(\Theta_n) = \mathcal{L}_n\big(E, \overline{\psi}\big) \in p^{k_\psi}\mathbb{Z}_p[\psi]$$

Proof. By assumption (E2) and [WW22, Theorem 2], the primitive algebraic L-value, defined as

$$\mathcal{L}\big(E, \overline{\psi}\big) := \frac{L(E, \overline{\psi}, 1)}{\varphi(m) e_\psi(\zeta_m) \Omega^{\psi(-1)}}$$

belongs to $\mathbb{Z}_p[\psi]$. It satisfies an explicit relation with $\mathcal{L}_n\big(E, \overline{\psi}\big)$, which can be expressed using the following formula

$$(-1)^{\nu(s)}\psi(s)\frac{\varphi(r)}{\varphi(n)} \prod_{\ell \in (S_n \cup \{p\}) \setminus S_m} \left(\frac{\ell - a_\ell \overline{\psi}(\ell) + \mathbf{1}_N(\ell)\overline{\psi}(\ell)^2}{\ell}\right) \mathcal{L}\big(E, \overline{\psi}\big)$$

When $r = n$, all of the terms in the right hand side belong to $\mathbb{Z}_p[\psi]$ with the possible exception of the Euler factor at p, which has p-adic valuation $k_{\overline{\psi}}$, which is the same as the one of k_ψ. Therefore,

$$\mathcal{L}_n\big(E, \overline{\psi}\big) \in p^{k_\psi}\mathbb{Z}_p[\psi] \qquad \square$$

We can now define the Stickelberger elements in the group algebra, encoding the different twisted algebraic L-values.

Definition 8.2.13. Let $\mathcal{D}_n$ be the set of Dirichlet characters modulo n. We define the *Stickelberger element* as

$$\Theta_n := \Theta_{\mathbb{Q}(\mu_n)} = \sum_{\psi \in \mathcal{D}_n} \mathcal{L}_n\big(E, \overline{\psi}\big) e_\psi \in \mathbb{Q}_p[\zeta_{\varphi(n)}][\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$$

The definition of the Stickelberger element can be extended to all abelian extensions.

Definition 8.2.14. Let $F/\mathbb{Q}$ be an abelian extension of conductor n. Then we define the Stickelberger element of F as

$$\Theta_F := N_{\mathbb{Q}(\mu_n)/F}\Theta_n$$

where $N_{\mathbb{Q}(\mu_n)/F} : \ \overline{\mathbb{Q}_p}[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})] \to \overline{\mathbb{Q}_p}[\mathrm{Gal}(F/\mathbb{Q})]$ is the norm map.

For some abelian extensions, the dual exponential of Kato's Euler system can be expressed as the Stickelberger element acting on a root of unit. However, that is only possible when the conductor is square-free. In general, we need to consider a combination of roots of unity.

Definition 8.2.15. Let $F/\mathbb{Q}$ be an abelian extension of conductor n. The *cyclotomic element* of F is defined as

$$\xi_F := \mathrm{Tr}_{\mathbb{Q}(\mu_n)/F}\left(\sum_{\tilde{n}|d|n} \zeta_d\right)$$

The main result of this section is shown below.

Theorem 8.2.16. The dual exponential of Kato's zeta elements satisfy the interpolative relation

$$w_F := (\exp^*_{\omega_E} \circ \mathrm{loc}^s_p)(z_F) = \Theta_F(\xi_F)$$

Before proving Theorem 8.2.16, we study how idempotent elements act on cyclotomic elements.

Lemma 8.2.17. If $F/\mathbb{Q}$ is an abelian extension of conductor n and ψ is a character of conductor m whose fixed field contains F, then

$$e_\psi(\xi_F) = [\mathbb{Q}(\mu_n) : F] e_\psi(\zeta_r)$$

Proof. For every d dividing n, let $d' = \gcd(d, m)$. For every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d'}))$, we can find a lift $\widetilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}(\mu_{d'}))$ such that

$$\widetilde{\sigma}\,|_{\mathbb{Q}(\mu_d)} = \sigma \text{ and } \quad \widetilde{\sigma}\,|_{\mathbb{Q}(\mu_m)} = \mathrm{Id}|_{\mathbb{Q}(\mu_m)}$$

Since ψ has conductor m,

$$e_\psi(\sigma(\zeta_d)) = e_\psi(\widetilde{\sigma}(\zeta_d)) = e_\psi(\zeta_d)\ \forall \sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d'}))$$

Therefore

$$e_\psi(\zeta_d) = \frac{1}{[\mathbb{Q}(\mu_d) : \mathbb{Q}(\mu_{d'})]} e_\psi(\mathrm{Tr}_{\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d'})}\zeta_d)$$

Assume first that there is a prime ℓ such that $v_\ell(d) > v_\ell(m) \geq 1$. Then $\gcd(d, m) \mid \frac{d}{\ell}$, so

$$\mathrm{Tr}_{\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d'})}\zeta_d = \mathrm{Tr}_{\mathbb{Q}(\mu_{d/\ell})/\mathbb{Q}(\mu_{d'})}\left(\mathrm{Tr}_{\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d/\ell})}\zeta_d\right)$$

However, $\mathrm{Tr}_{\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu_{d/\ell})}\zeta_d = 0$ because $\ell \mid \frac{d}{\ell}$. Indeed, ζ_d is a root of the polynomial $P(T) = T^\ell - \zeta_{d/\ell}$. Since $[\mathbb{Q}(\mu_d) : \mathbb{Q}(\mu_{d/\ell})] = \ell$ because $\ell^2 \mid d$, then $P(T)$ is irreducible. The sum of the roots of P is zero and, hence, so is the trace of ζ_d. Therefore, $e_\psi(\zeta_d) = 0$ in this case.

Given some d such that the above prime ℓ does not exist, then $d \mid r$. Moreover, assume $d < r$. Since $\tilde{n} \mid d$, then $d' < m$

$$e_\psi(\zeta_d) = \frac{1}{[\mathbb{Q}(\mu_d) : \mathbb{Q}(\mu_{d'})]} e_\psi(\mathrm{Tr}_{\mathbb{Q}(\mu_d)/\mathbb{Q}(\mu'_d)}\zeta_d)$$

Since ψ has conductor m, then $e_\psi(x) = 0$ for every $x \in \mathbb{Q}(\mu_{d'})$ and, therefore, $e_\psi(\zeta_d) = 0$.

In the following formula, the only remaining term is ζ_r, so

$$e_\psi(\xi_{\mathbb{Q}(\mu_n)}) = \sum_{\tilde{n}|d|n} e_\psi(\zeta_d) = e_\psi(\zeta_r)$$

Since F contains the fixed field of ψ, by lemma 8.2.17,

$$e_\psi(\xi_F) = [\mathbb{Q}(\mu_n) : F]e_\psi(\xi_n) = [\mathbb{Q}(\mu_n) : F]e_\psi(\zeta_r) \qquad \square$$

Proof of Theorem 8.2.16. We can compute

$$e_\psi(\Theta_F(\xi_F)) = \Theta_n e_\psi(\xi_F) = [\mathbb{Q}(\mu_n) : F]\mathcal{L}_n\big(E, \overline{\psi}\big)e_\psi(\zeta_r)$$

Thus,

$$e_\psi(\Theta_F(\xi_F)) = \frac{L_{S_n \cup \{p\}}(E, \overline{\psi}, 1)}{[F : \mathbb{Q}]\Omega^{\psi(-1)}} \in F \otimes \mathbb{Q}_p$$

Since that is true for all characters, we can conclude from equation (8.7) that

$$w_F = \Theta_F(\xi_F) \in F \otimes \mathbb{Q}_p \qquad \square$$

8.2.3 Twisting of Kato's Euler system

We are interested in studying the representation

$$T(\chi) := T_pE \otimes \mathcal{O}_d(\chi)$$

where χ is a primitive Dirichlet character of $G_{\mathbb{Q}}$ whose image has order dividing d.

There are two equivalent ways of constructing an Euler system for this Galois representation, both building from Kato's construction (see §A.3.2). As shown in Definition 8.1.16, we can see $T(\chi)$ both as the twisted representation of the Tate module of an elliptic curve or as the Galois representation associated with a modular form.

We start by considering $T(\chi)$ as the twist of the Tate module by a Dirichlet character. For the Tate module, there is a canonical Kato's Euler system that interpolates the L-values normalised by Néron periods (see Theorem A.3.2). From this Euler system, we can construct another Euler system for $T(\chi)$ by applying a general twisting process of Euler systems described in [Rub00, §II.4].

Definition 8.2.18. Let T be a Galois representation of $G_{\mathbb{Q}}$ and let χ be a Dirichlet character with fixed field K and conductor dividing d. Then we can defined *twisted Euler system* by

$$\mathbf{c}_\chi = c_{\chi,F} := \mathrm{cor}_{KF/F}(c_{KF} \otimes 1_\chi)$$

where 1_χ represents the unit element in $\mathcal{O}_d(\chi)$.

Proposition 8.2.19. ([Rub00, proposition II.4.2]) The collection $\{c_{F,\chi}\}_{F\in\Omega}$ is an Euler system for $T(\chi)$.

Remark 8.2.20. In Theorem A.3.2, a canonical Euler system $\mathbf{z} = \{z_F\}_{F\in\Omega}$ in was constructed. By Proposition 8.2.19, the collection $\mathbf{z}_\chi = \{z_{F,\chi}\}_{F\in\Omega}$ obtained after applying the twisting process in Definition 8.2.18 is an Euler system in $\mathrm{ES}(T(\chi))$.

However, $T(\chi)$ can be also seen as the Galois representation associated with the twisted modular form f_χ, defined in Definition 8.1.16. For this representation, there is a collection of Euler systems constructed in Theorem A.3.5. We will show below that the twisted Euler system $\mathbf{z}_\chi$ is the Euler system constructed in Theorem A.3.5 for a suitable choice of $\gamma \in V_{\mathbb{C}}(f_\chi)$.

We follow the argument in [Kat04, §14.6]. There is an isomorphism of Galois representations

$$\Psi : \; V_{\mathbb{Q}_p}(f) \otimes L_\lambda(\chi) \cong V_{L_\lambda}(f_\chi) \tag{8.8}$$

Choose $\gamma = \Psi(\gamma_0 \otimes 1)$. Note that $\gamma^{\pm\chi(-1)} = \Psi(\gamma^{\pm} \otimes 1)$. The isomorphism in (8.8) induces an isomorphism of cohomology groups:

$$H^1(F, V_{\mathbb{Q}_p}(f) \otimes L_\lambda(\chi)) \cong H^1(F, V_{L_\lambda}(f_\chi)) \tag{8.9}$$

Choose γ the image of $\gamma_0 \otimes 1$ under the isomorphism

We claim that $z_{\gamma,F}^{f_\chi}$ is the image of $z_{\chi,F}^{f}$ under the isomorphism in (8.9). Consider the commutative diagram

$$\begin{tikzcd}
H^1(F, V_{\mathbb{Q}_p}(f) \otimes L_\lambda(\chi)) \arrow[r, "\exp^*"] \arrow[d, "\sim"] & S(f) \otimes L_\lambda(\chi) \arrow[r, "\mathrm{per}"] & V_{\mathbb{C}}(f) \arrow[d, "\sim"] \\
H^1(F, V_{L_\lambda}(f_\chi)) \arrow[r, "\exp^*"] & S(f_\chi) \otimes L_\lambda \arrow[r, "\mathrm{per}"] & V_{\mathbb{C}}(f_\chi)
\end{tikzcd}$$

Following [Kat21, (6.3)], we can compute the image of the zeta elements under the composition $\mathrm{per} \circ \exp^*$. If ψ is a character of $\mathrm{Gal}(F/\mathbb{Q})$, we have that

$$\begin{aligned}
(\mathrm{per} \circ \exp^*)\Big(e_\psi\big(z_{F,\chi}^{f}\big)\Big) &= L_{S_F \cup \{p\}}(E, \chi\psi, 1)(\gamma_0^{\chi\psi(-1)} \otimes 1_{\mathbb{C}}) \\
(\mathrm{per} \circ \exp^*)\Big(e_\psi\big(z_{\gamma,F}^{f_\chi}\big)\Big) &= L_{S_F \cup \{p\}}(f_\chi, \psi, 1)(\gamma^{\psi(-1)} \otimes 1_{\mathbb{C}})
\end{aligned}$$

Since $L_{S_F \cup \{p\}}(E, \chi\psi) = L_{S_F \cup \{p\}}(f_\chi, \psi)$ and $\gamma_0^{\chi\psi(-1)} \otimes 1_{\mathbb{C}}$ is identified with $\gamma^{\psi(-1)} \otimes 1_{\mathbb{C}}$ under the isomorphism on the right, the images of both zeta elements are the same under the identifications made. Since the compositions $\mathrm{per} \circ \exp^*$ are injective maps, we can conclude that both zeta elements are identified under the isomorphism

$$H^1(F, V_{\mathbb{Q}_p}(f) \otimes L_\lambda(\chi)) \cong H^1(F, V_{L_\lambda}(f_\chi))$$

8.2.4 Twisted dual exponential map

In §8.2.1, we have studied the dual exponential map for the Galois representation $V := T_pE \otimes \mathbb{Q}_p$. However, we have seen in (8.1) that the representation of interest is a Galois invariant lattice in the twist $V \otimes \mathcal{O}_d(\chi)$, where χ is a Dirichlet character. In this section, we will give a description of the dual exponential map of $V \otimes \mathcal{O}_d(\chi)$.

Remark 8.2.21. Note that V and $V \otimes \mathcal{O}_d(\overline{\chi})$ are equal as $G_{(KF)_w}$-modules for every prime w of KF above p. Hence, the Néron differential also induces a dual exponential map for the twisted representation over the field (KF):

$$\bigoplus_{w|p} \frac{(B_{\mathrm{dR}} \otimes V \otimes \mathcal{O}_d(\overline{\chi}))^{G_{(KF)_w}}}{(B_{\mathrm{dR}}^+ \otimes V \otimes \mathcal{O}_d(\overline{\chi}))^{G_{(KF)_w}}} \to \bigoplus_{w|p} H^1_{\mathcal{F}_{\mathrm{BK}}}((KF)_w, V \otimes \mathcal{O}_d(\overline{\chi}))$$

For every w, we have an isomorphism (depending on fixing a Weierstrass model for E)

$$\frac{(B_{\mathrm{dR}} \otimes V \otimes \mathcal{O}_d(\overline{\chi}))^{G_{(KF)_w}}}{(B_{\mathrm{dR}}^+ \otimes V \otimes \mathcal{O}_d(\overline{\chi}))^{G_{(KF)_w}}} \cong (KF)_w \otimes \mathcal{O}_d(\overline{\chi})$$

Since $G_{w/v} := \mathrm{Gal}((KF)_w/F_v)$ is finite, then

$$H^1\left((KF)_w/F_v, B_{\mathrm{dR}}^+ \otimes V \otimes \overline{\chi}\right) = 0$$

Then,

$$\left(\frac{\left(B_{\mathrm{dR}} \otimes V \otimes \mathcal{O}_d(\overline{\chi})\right)^{G_{(KF)_w}}}{\left(B_{\mathrm{dR}}^+ \otimes V \otimes \mathcal{O}_d(\overline{\chi})\right)^{G_{(KF)_w}}}\right)^{G_w} = \frac{\left(B_{\mathrm{dR}} \otimes V \otimes \mathcal{O}_d(\overline{\chi})\right)^{G_{F_v}}}{\left(B_{\mathrm{dR}}^+ \otimes V \otimes \mathcal{O}_d(\overline{\chi})\right)^{G_{F_v}}}$$

By considering the direct sum over all $w \mid p$, the exponential map over F_v can be written as

$$\exp_{\omega_E,\overline{\chi}} : \left(KF \otimes \mathbb{Q}_p \otimes \mathcal{O}_d(\overline{\chi})\right)^{\mathrm{Gal}(KF/F)} \to \bigoplus_{v|p} H^1_{\mathcal{F}_{\mathrm{BK}}}\left((F_v, V \otimes \mathcal{O}_d(\overline{\chi}))\right)$$

Note that the first term is the χ-part of $KF \otimes \mathbb{Q}_p$. Hence the dual exponential map can be written as

$$\exp^*_{\omega_E,\overline{\chi}} : \bigoplus_{v|p} H^1_{/\mathcal{F}_{\mathrm{BK}}}(F_v, V \otimes \mathcal{O}_d(\chi)) \to \mathrm{Hom}\big(e_\chi(KF \otimes L), \mathbb{Q}_p\big)$$

where we denote $L = \mathcal{O}_d \otimes \mathbb{Q}_p$. Since the identification in (8.5) is Galois equivariant, and identifying $\mathrm{Hom}(L, \mathbb{Q}_p) \cong L$ using (8.5), we obtain that

$$\mathrm{Hom}\big(e_\chi(KF \otimes L), \mathbb{Q}_p\big) \cong e_{\overline{\chi}}\big((KF) \otimes L\big)$$

In an abuse of notation, we will also denote by $\exp^*_{\omega_E,\overline{\chi}}$ the following composition

$$\exp^*_{\omega_E,\overline{\chi}} \circ \mathrm{loc}^s_p : \ H^1(F, V \otimes \mathcal{O}_d(\chi)) \to e_{\overline{\chi}}((KF) \otimes L)$$

Now we will describe how the twisting process in Proposition8.2.19 is reflected in the images of the dual exponential map.

First note that over KF, the exponential map $\exp^*_{\omega_E,\overline{\chi}}$ coincides with $\exp^*_{\omega_E} \otimes \mathcal{O}_d(\chi)$. Hence we just need to see how $\exp^*_{\omega_E,\overline{\chi}}$ behaves under the corestriction.

Proposition 8.2.22. Let $c \in H^1(F, V \otimes \mathcal{O}_d(\chi))$ and consider its corestriction $d = \operatorname{cor}_{KF/F} c \in H^1(KF, V \otimes \mathcal{O}_d(\chi))$. Then

$$\exp^*_{\omega_E,\overline{\chi}}(d) = N_{KF/F} \exp^*_{\omega_E,\overline{\chi}}(c) \in (KF \otimes \mathbb{Q}_p \otimes \mathcal{O}_d(\chi))^{G_F}$$

Proof. Localising at primes above p, we have that

$$\operatorname{loc}^s_p(d) = \left(\bigoplus_{w|p} \operatorname{cor}_{(KF)_w/F_v} \right) (\operatorname{loc}^s_p(c))$$

By [NSW00, Proposition 1.5.3 (iv)], if we understand $\operatorname{loc}^s_p(c)^\vee$ and $\operatorname{loc}^s_p(d)^\vee$ as maps into the duals of the finite cohomology groups, we have that

$$\operatorname{loc}^s_p(c)^\vee = \operatorname{loc}^s_p(d)^\vee \circ \left(\bigoplus_{w|p} \operatorname{res}_{(KF)_w/F_v} \right)$$

By [NSW00, Proposition 1.5.2], the following diagram is commutative

$$\begin{tikzcd}
(KF \otimes \mathbb{Q}_p \otimes \overline{\chi})^{G_F} \arrow[r, "\exp_{\omega_E}"] \arrow[d, "\subset"] & \bigoplus_{v|p} H^1_{\mathrm{f}}(F_v, V \otimes \overline{\chi}) \arrow[r, "\operatorname{loc}^s_p(d)^\vee"] \arrow[d, "\bigoplus_{v|p} \operatorname{res}"] & \mathbb{Q}_p \arrow[d, "="] \\
KF \otimes \mathbb{Q}_p \otimes \overline{\chi} \arrow[r, "\exp_{\omega_E}"] & \bigoplus_{w|p} H^1_{\mathrm{f}}((KF)_w, V \otimes \overline{\chi}) \arrow[r, "\operatorname{loc}^s_p(c)^\vee"] & \mathbb{Q}_p
\end{tikzcd}$$

Therefore, $\exp^*_{\omega_E,\chi}(d)$ is the restriction to $(KF \otimes \mathbb{Q}_p \otimes \overline{\chi})^{G_F}$ of $\exp^*_{\omega_E,\chi}(c)$. Under the identification (8.5), that means

$$\exp^*_{\omega_E,\chi}(d) = N_{KF/F} \exp^*_{\omega_E,\chi}(c) \qquad \square$$

Notation 8.2.23. We denote

$$w_{F,\chi} := \exp^*_{\omega_E,\chi}(z_{F,\chi})$$

Theorem 8.2.24. When $K \cap F = \mathbb{Q}$, the exponential map of the twisted Kato's Euler system can be computed as

$$\omega_{F,\chi} = de_{\overline{\chi}}(\Theta_{KF})(\xi_{KF})$$

where, recall, d was the degree $[K : \mathbb{Q}]$, Θ_{KF} was the Stickelberger element constructed in Definition 8.2.14 and ξ_F is the cyclotomic element defined in Definition 8.2.15.

Proof. By Proposition 8.2.22, the dual exponential map of the twisted Kato's Euler system is

$$\exp^*_{\omega_E,\overline{\chi}}(z_{F,\chi}) = N_{KF/F}(\exp^*_{\omega_E,\overline{\chi}}(z_{KF} \otimes 1_\chi)) = [K : \mathbb{Q}]e_{\overline{\chi}}(\omega_{KF})$$

By Theorem 8.2.16,

$$w_{F,\chi} = [K : \mathbb{Q}]e_{\overline{\chi}}(\Theta_{KF}(\xi_{KF})) = d\, e_{\overline{\chi}}(\Theta_{KF})(\xi_{KF}) \qquad \square$$

8.2.5 Image of the dual exponential map

In §8.2.1, we have introduced the dual exponential map

$$\exp^*_{\omega_E} : \ H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, V) \xrightarrow{\sim} K_\mathfrak{p}$$

where $K_\mathfrak{p}$ is the completion of K at a prime $\mathfrak{p}$ above p. However, the group we are interested in is $H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, T)$. Hence we want to know the image of the composition map

$$\exp^*_{\omega_E} : \ H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, T) \to H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, V) \to K_\mathfrak{p}$$

Theorem 8.2.25. The image to the twisted dual exponential map is

$$\exp^*_{\omega_E}\big(H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, T)\big) = p^{k_{\overline{\chi}}}(\mathcal{O}_p)_{\overline{\chi}}$$

where k_χ is the quantity defined in Notation 8.2.11.

Proof. By the identifications made so far, some $z \in H^1_{/\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, T)$ is identified under local Tate duality to the map

$$E(K_\mathfrak{p}) \otimes \mathbb{Q}_p \to \mathbb{Q}_p, \ x \mapsto \mathrm{Tr}_{K_\mathfrak{p}/\mathbb{Q}_p}(\exp^*_{\omega_E}(z) \log_{\omega_E}(x))$$

Hence an element $y \in K_\mathfrak{p}$ belongs to $\exp^*_{\omega_E}(H^1_{\mathcal{F}_{\mathrm{BK}}}(K_\mathfrak{p}, T))$ if and only if

$$\mathrm{Tr}_{K_\mathfrak{p}/\mathbb{Q}_p}(y \log_{\omega_E}(x)) \in \mathbb{Z}_p \ \forall x \in E(K_\mathfrak{p}) \tag{8.10}$$

where the logarithm is the extension of the one defined in the formal group of the elliptic curve.

Denote by $\mathcal{O}_\mathfrak{p}$ and $\mathfrak{m}_\mathfrak{p}$ to the ring of integers of $K_\mathfrak{p}$ and its maximal ideal, respectively. Denote by $E_1(K_\mathfrak{p})$ the kernel of the reduction map and $E_0(K_\mathfrak{p})$ the points whose reduction is a non-singular point. Since $K_\mathfrak{p}/\mathbb{Q}_p$ is unramified by (K1), the logarithm induces an isomorphism

$$\log_{\omega_E} : E_1(K_\mathfrak{p}) \xrightarrow{\sim} \mathfrak{m}_\mathfrak{p}$$

To describe the image $\log_{\omega_E}(E(K_\mathfrak{p}))$, we look at the χ-part of this map for the different characters χ of $\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)$. By (K3), $E(K_\mathfrak{p})_\chi$ is free of rank one over $\mathcal{O}[\chi]$. Since p does not divide the Tamagawa number $c_\mathfrak{p}$ by (K4), then the quotient $E(K_\mathfrak{p})/E_0(K_\mathfrak{p})$ has order prime to p. Therefore

$$\log_{\omega_E}(E(K_\mathfrak{p})) = \log_{\omega_E}(E_0(K_\mathfrak{p}))$$

Consider the exact sequence

$$0 \longrightarrow E_1(K_\mathfrak{p}) \longrightarrow E_0(K_\mathfrak{p}) \longrightarrow \widetilde{E}_0(\kappa_\mathfrak{p}) \longrightarrow 0$$

where $\widetilde{E}_0(\kappa_\mathfrak{p})$ denote the groups of non-singular points of the reduced curve modulo $\mathfrak{p}$. Since $[K_\mathfrak{p} : \mathbb{Q}_p]$ is prime to p by (K2), the sequence remains exact after taking χ-parts. Since $E(K_\mathfrak{p})[p] = 0$ by (K3), we have that

$$\log_{\omega_E}(E_0(K_p)_\chi) = \frac{1}{p^{\mathrm{length}(\widetilde{E}_0(\kappa_\mathfrak{p})[p^\infty]_\chi)}} \log_{\omega_E}(E_1(K_p)_\chi)$$

Since $K_\mathfrak{p}/\mathbb{Q}_p$ is unramified by (K1), then $\log_{\omega_E}(E_1(K_\mathfrak{p})_\chi) = (\mathfrak{m}_\mathfrak{p})_\chi = p(\mathcal{O}_\mathfrak{p})_\chi$ and

$$\log_{\omega_E}(E(K_\mathfrak{p})_\chi) = \frac{p}{p^{\mathrm{length}(\widetilde{E}_0(\kappa_\mathfrak{p})[p^\infty]_\chi)}} \mathcal{O}_\chi \tag{8.11}$$

The trace map satisfies, for every $x \in K_\mathfrak{p}$, the identity $\mathrm{Tr}(x) = \mathrm{Tr}(e_1 x)$, where $e_1 \in \mathbb{Z}_p[\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)]$ is the idempotent element associated with the trivial character, i.e., $e_1 = \frac{1}{[K_\mathfrak{p}:\mathbb{Q}_p]} \sum_{\sigma \in \mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)} \sigma$. By (K2), $[K_\mathfrak{p} : \mathbb{Q}_p]$ is prime to p. Thus $\mathrm{Tr}(x) \in \mathbb{Z}_p$ if and only if $e_1 x \in \mathbb{Z}_p$.

Moreover, note that given $x_1, x_2 \in K_\mathfrak{p}$ such that $\sigma(x_1) = \chi_1(\sigma)x_1$ and $\sigma(x_2) = \chi_2(\sigma)x_2$ for every $\sigma \in \mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)$ and some characters χ_1 and χ_2, then $\sigma(x_1 x_2) = (\chi_1\chi_2)(\sigma)(x_1 x_2)$.

Combining equations (8.10) and (8.11), we obtain

$$\exp^*(H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T \otimes \mathcal{O}_d(\chi))) = p^{\mathrm{length}(\widetilde{E}_0(\kappa_\mathfrak{p})[p^\infty]_{\overline{\chi}})-1} \mathcal{O}_{\overline{\chi}} \tag{8.12}$$

Then the proof of the theorem is concluded by Lemma 8.2.26 below. □

Lemma 8.2.26. The length of $e_\chi\left(\widetilde{E}_0(\kappa_p)[p^\infty]\right)$ as an $\mathcal{O}_d$-module is one unit larger than the valuation of the twsited Euler factor $1-\frac{a_p}{p}\chi(p)+\mathbf{1}_N(p)\frac{1}{p}\chi(p)^2$ at p evaluated at $s=1$.

Proof. Recall the definition of $k_\chi = v_p\left(1-\frac{a_p}{p}\chi(p)+\mathbf{1}_N(\ell)\frac{1}{p}\chi(p)^2\right)$ in definition 8.1.22. We will consider different cases depending of the type of reduction of E at p.

Assume first that E has good ordinary reduction at p and let $\alpha\in\mathbb{Z}_p^\times$ be the unit root of the Euler polynomial X^2-a_pX+p. Then the arithmetic Frobenius acts on the reduced p-primary torsion $\widetilde{E}[p^\infty]$ by multiplication by α and thus acts on $\widetilde{E}[p^\infty]\otimes O(\overline{\chi})$ multiplying by $\alpha\overline{\chi}(p)$. $\widetilde{E}(\kappa_\mathfrak{p})_\chi$ is the kernel of the action of Frob_p-1 on $\widetilde{E}[p^\infty]\otimes O(\overline{\chi})$, so its length is the p-adic valuation of $(\alpha\overline{\chi}(p)-1)$.

Thus we just need to compute the p-adic valuation of $\alpha\overline{\chi}(p)-1$. The twisted Euler polynomial factors as

$$\chi(p)^2X^2-a_p\chi(p)X+p=\chi(p)^2(\overline{\chi}(p)\alpha-X)(\overline{\chi}(p)\beta-X)$$

where $\beta\in p\mathbb{Z}_p$ is the other root. Since $\overline{\chi}(p)\beta-1$ is a unit, evaluating at $X=1$ we get

$$(\chi(p)^2-\chi(p)a_p+p)\sim(\overline{\chi}(p)\alpha-1)$$

where $\sim$ denotes equality up to multiplication by a p-adic unit.

It E has good supersingular reduction at p, then clearly $\widetilde{E}(\kappa_\mathfrak{p})[p^\infty]=\{O\}$. In the supersingular case, then $p\nmid N$ and $p\mid a_p$, so the k_χ has p-adic valuation -1 when evaluated at $X=1$. Since $E(\kappa_\mathfrak{p})_\chi=\{0\}$, then the proposition holds true in this case.

If E has multiplicative reduction, then $\widetilde{E}_0(\kappa_\mathfrak{p})\cong\kappa_\mathfrak{p}^\times$ has order prime to p, so the p-adic valuation of the right hand side is -1. In this case, $a_p=\pm1$ depending on the reduction being split or not and $\mathbf{1}_N(\ell)=0$, so k_χ has p-adic valuation -1 as well.

If E has additive reduction, then

$$\widetilde{E}_0(\kappa_\mathfrak{p})_\chi\cong(\kappa_\mathfrak{p})_\chi=\#(\mathcal{O}_\mathfrak{p}/p\mathcal{O}_p)_\chi$$

has length one, so the equality is also satisfied in this case. $\square$

We can then use the exponential map to construct isomorphisms from the finite quotients of T.

Corollary 8.2.27. The dual exponential map induces an isomorphism

$$\exp^*_{\omega_E}:\ H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T/p^kT\otimes\mathcal{O}_d(\chi)) \to \frac{p^{k_{\overline{\chi}}}(\mathcal{O}_p)_{\overline{\chi}}}{p^{k+k_{\overline{\chi}}}(\mathcal{O}_p)_{\overline{\chi}}}$$

Proof. By (K3) in Assumption (Ab), we have that

$$H^2(\mathbb{Q}_p, T\otimes\mathcal{O}_d(\chi)) = H^0(\mathbb{Q}_p, E[p^\infty]\otimes\mathcal{O}_d(\overline{\chi})) = 0$$

Then there is an isomorphism

$$H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T\otimes\mathcal{O}_d(\chi))/p^k \cong H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T/p^kT\otimes\mathcal{O}_d(\chi))$$

The dual exponential map induces thus an isomorphism

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T/p^kT\otimes\mathcal{O}_d(\chi)) \cong \frac{\exp^*_{\omega_E}(H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T\otimes\mathcal{O}_d(\chi)))}{p^k\exp^*_{\omega_E}(H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T\otimes\mathcal{O}_d(\chi)))}$$

By Theorem 8.2.25, there is another isomorphism

$$\exp^*_{\omega_E}:\ H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T/p^kT\otimes\mathcal{O}_d(\chi)) = \frac{p^{k_{\overline{\chi}}}(\mathcal{O}_p)_{\overline{\chi}}}{p^{k+k_{\overline{\chi}}}(\mathcal{O}_p)_{\overline{\chi}}}$$

where $\mathcal{O}_p := \bigoplus_{\mathfrak{p}|p}\mathcal{O}_{\mathfrak{p}} \subset K\otimes\mathbb{Q}_p$. □

8.3 The Kolyvagin system

8.3.1 Mazur-Tate elements

Using the interpolation in Theorem 8.2.16, we can relate ω_F to Mazur-Tate elements defined in [MT87]. The main advantage of this relation is that Mazur-Tate elements have an explicit formula in terms of the modular symbols defined in Definition 8.1.12, which is a key fact in the relation between Kato's Euler system and the Kurihara numbers.

Definition 8.3.1. Let $n\in\mathbb{Z}$. The *Mazur-Tate modular element* for n is defined as

$$\theta_n = \sum_{a\in(\mathbb{Z}/n\mathbb{Z})^\times}\left(\left[\frac{a}{n}\right]^+ + \left[\frac{a}{n}\right]^-\right)\sigma_a \in \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$$

where σ_a is the element of $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ that sends ζ_n to ζ_n^a. The integrality of the Mazur-Tate modular elements holds true under assumption (E2).

Analogously to what happened with Stickelberger elements, Mazur-Tate elements can be also defined for abelian extensions of $\mathbb{Q}$.

Definition 8.3.2. Let $F/\mathbb{Q}$ be an abelian extension of conductor n. Then the Mazur-Tate element of F is defined as

$$\theta_F := c_{\mathbb{Q}(\mu_n)/F}\theta_n$$

Mazur-Tate elements can be related to special L-values by using the Birch's formula (see [MTT86, formula (8.6)])

Proposition 8.3.3. ([MT87, §1.4],[WW22, Lemma 6, Proposition 7]) If ψ is a Dirichlet character of conductor n, then

$$\psi(\theta_n) = \frac{n}{\varphi(n)e_\psi(\zeta_n)} \frac{L_{S_n}(E, \overline{\psi}, 1)}{\Omega^{\psi(-1)}} \in \mathbb{Z}_p[\psi]$$

where S_n is the set of primes dividing n and $\varphi(n)$ is the Euler totient function. Note that the product $\varphi(n)e_\psi(\zeta_n)$ coincides with the Gauss sum $\tau(\overline{\psi})$.

The goal of this section is to compare Mazur-Tate elements with the Stickelberger elements constructed in Definitions 8.2.13 and 8.2.14. Proposition 8.3.3 allows us to compare the ψ parts of Θ_n and θ_n when the ψ is a character of conductor n. However, that comparison cannot be done for characters whose conductor is strictly smaller that n.

Corollary 8.3.4. If ψ is a Dirichlet character of conductor n, then

$$\psi(\Theta_n) = \frac{1}{n}\psi(\theta_n)(1 - p^{-1}a_p\overline{\psi}(p) - p^{-1}\mathbf{1}_N(p)\overline{\psi}(p)^2)$$

Remark 8.3.5. For every $n \in \mathcal{N}_k$, recall that $\mathbb{Q}(n)$ is the maximal p-subextension inside $\mathbb{Q}(\mu_n)$ and let $K(n) := K\mathbb{Q}(n)$. Note that there is a canonical identification

$$\mathrm{Gal}(K(n)/\mathbb{Q}) = \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(n)/\mathbb{Q}) = G \times \mathcal{G}_n$$

We can use Corollary 8.3.4 to compare $\Theta_{K(n)}$ and $\theta_{K(n)}$ as elements in the group ring $\mathbb{Z}_p[\mathrm{Gal}(K(n)/\mathbb{Q})]$. We follow the process in [Ota18] and [KKS20]. However, in our case, we only have the equality for the primitive character parts. That implies that both elements are not necessarily equal, but they are related enough so we can compare their Kolyvagin derivatives in the next section.

Definition 8.3.6. Consider the element in the group algebra $\mathbb{Z}_p[G]$ defined as

$$\Upsilon_{K(n)} := \Theta_K(n) - \frac{1}{n}\big(1 - p^{-1}a_p\mathrm{Frob}_p^{-1} - p^{-1}\mathbf{1}_N(p)\mathrm{Frob}_p^{-2}\big)\theta_{K(n)}$$

The following result can be deduced from Corollary 8.3.4

Corollary 8.3.7. For every primitive character ψ of $\mathrm{Gal}(K(n)/\mathbb{Q})$, we have that [2]

$$e_\psi \Upsilon_{K(n)} = 0$$

Using the decomposition in Remark 8.3.5, we can decompose characters of $\mathrm{Gal}(K(n)/\mathbb{Q})$ as products of a character of G and character of $\mathcal{G}_n$.

Recall that χ was a primitive character of $\mathrm{Gal}(K/\mathbb{Q})$ which was used to construct $T(\chi)$. Using the identification in Remark 8.3.5, we can consider $\chi(\Theta_n)$ and $\chi(\theta_n)$ as elements in $\mathbb{Z}_p[\chi][\mathcal{G}_n]$.

Corollary 8.3.8. Let χ be a primitive character of $\mathrm{Gal}(K/\mathbb{Q})$. Then

$$\chi\left(\Upsilon_{K(n)}\right) = \sum_{\ell | n} \nu_{n,n/\ell}\ \alpha_{n,\ell}$$

where $\nu_{n,n\ell} :\ \mathbb{Q}_p[\zeta_{cn}][\mathcal{G}_{n/\ell}] \to \mathbb{Q}_p[\zeta_{cn}][\mathcal{G}_n]$ is the norm map and $\alpha_{n,\ell}$ is certain element in $p^{k_\chi}\mathbb{Z}_p[\zeta_{cn}][\mathcal{G}_{n/\ell}]$.

Proof. For every Dirichlet character ψ of conductor n, then $\chi \times \psi$ is a primitive character of $\mathrm{Gal}(K(n)/\mathbb{Q})$ and we have that

$$e_{\chi\times\psi}\Upsilon_{K(n)} = 0$$

For every character ψ of $\mathrm{Gal}(\mathbb{Q}(n)/\mathbb{Q})$ (not necessarily primitive), $\chi \times \psi$ has conductor m satisfying that $r(m, cn) = cn$. Therefore, Proposition 8.2.12 implies that $\chi(\Theta_{K(n)}) \in p^{k_\chi}\mathbb{Z}_p[\mathcal{G}_n]$. Hence $\alpha_{n,\ell} \in p^{k_\chi}\mathbb{Z}_p[\zeta_{cn}][\mathcal{G}_{n/\ell}]$ for every prime divisor ℓ of n. □

8.3.2 Kolyvagin derivatives

In this section, we will use the relation appearing in Corollary 8.3.8 to compare the Kolyvagin derivatives of the (primitive) character

[2]Here, primitive is understood as having the same conductor as the abelian extension $K/\mathbb{Q}(n)$.

parts of $\Theta_{K(n)}$ and $\theta_{K(n)}$. This comparison will then lead to a computational description of the Kolyvagin derivative of the Stickelberger element in terms of the Kurihara numbers.

Proposition 8.3.9. Let χ be a primitive character of G and let $n \in \mathcal{N}_{k},$. Then the Kolyvagin derivative, constructed in Definitions A.4.1 and A.4.4 can be computed, modulo $p^{k+k_\chi}\mathbb{Z}_p[\chi][\mathcal{G}_n]$, as

$$D_n\chi(\Theta_{K(n)}) \equiv \frac{1}{n}\left(1 - p^{-1}a_p\overline{\chi}(p) + p^{-1}\mathbf{1}_N(p)\overline{\chi}(p)^2\right) D_n\chi(\theta_{K(n)})$$

Proof. Note that, since $\mathrm{Gal}(K(n)/\mathbb{Q})$ is abelian,

$$\begin{aligned}&D_n\left(\chi\left((1 - p^{-1}a_p\mathrm{Frob}_p^{-1} + p^{-1}\mathbf{1}_N(p)\mathrm{Frob}_p^{-2})\theta_{K(n)}\right)\right) = \\ &\chi\left(1 - p^{-1}a_p\mathrm{Frob}_p^{-1} - p^{-1}\mathrm{Frob}_p^{-2}\right) D_n(\chi(\theta_{K(n)})) = \\ &(1 - p^{-1}a_p\overline{\chi}(p) + p^{-1}\mathbf{1}_N(p)\overline{\chi}(p)^2)D_n(\chi(\theta_{K(n)}))\end{aligned}$$

By Corollary 8.3.8, it is enough to prove that, for every prime divisor ℓ of n,

$$D_n\,\nu_{n,n/\ell}\alpha \in p^{k+k_\chi}\mathbb{Z}_p[\psi][\mathcal{G}_n]$$

for every $\alpha \in p^{k_\chi}\mathbb{Z}_p[\psi][\mathcal{G}_{n/\ell}]$. In fact, since $\nu_{n,n/\ell}\alpha$ is $\mathcal{G}_\ell$ invariant, then

$$D_\ell\nu_{n,n/\ell}\alpha = \frac{p^{n_\ell}(p^{n_\ell}-1)}{2}\nu_{n,n/\ell}\alpha \in p^{k+k_\chi}\mathbb{Z}_p[\psi][\mathcal{G}_n]$$

since $\ell \in \mathcal{P}_k$. Thus

$$D_n\nu_{n,n/\ell}\alpha = D_{n/\ell}D_\ell\nu_{n,n/\ell}\alpha \in p^{k+k_\chi}\mathbb{Z}_p[\psi][\mathcal{G}_n]$$ □

By Proposition A.4.6, for every $n \in \mathcal{N}_k$, $D_n z_{K\mathbb{Q}(n)}$ is invariant under the action of $\mathcal{G}_n$ modulo p^k. Consequently, $D_n\Theta_{K\mathbb{Q}(n)}$ and, therefore, $D_n\theta_{K\mathbb{Q}(n)}$ are $\mathcal{G}_n$-invariants modulo p^k. This is equivalent to

$$D_n\Theta_{K(n)}(\zeta_{K(n)}),\ D_n\theta_{K(n)}(\zeta_{K(n)}) \in K \otimes \mathbb{Q}_p$$

In order to compute this value, Proposition 8.3.12 below is very useful. Before stating it, we need to define a p-primary logarithm in $(\mathbb{Z}/\ell)^\times$.

Definition 8.3.10. Assume H is a finite cyclic group whose order is exactly divisible by p^k for some $k \in \mathbb{N}$. If x is a generator of the p-primary part H, then for every $a \in H$ we will define $\log_x(a)$ to be the unique element in $y \in \mathbb{Z}/p^k$ such that $a^{-1}x^y$ has order prime to p.

Remark 8.3.11. Given two generators x_1 and x_2 of the p-primary part of H, the logarithms $\log_{x_1}(a)$ and $\log_{x_2}(a)$ have the same p-adic valuation.

Proposition 8.3.12. ([Sak22, lemma 3.11]) Let R be a ring, let $n = \ell_1 \cdots \ell_s \in \mathcal{N}_k$ and let

$$\theta = \sum_{\sigma \in \mathcal{G}_n} a_\sigma \sigma \in R[\mathcal{G}_n]$$

be an element such that $D_n\theta$ is Galois invariant modulo p^k. Then

$$D_n\theta \equiv \sum_{\sigma \in \mathcal{G}_n} a_\sigma \prod_{\ell | n} \log_{\tau_\ell}(\sigma) N_n \mod p^k R[\mathcal{G}_n]$$

where $N_n = \sum_{\sigma \in \mathcal{G}_n} \sigma$ is the norm element and τ_ℓ is the generator of $\mathcal{G}_\ell$ used to define the Kolyvagin derivative.

Proof. Denoting $T_i = \tau_{\ell_i} - 1$ for every i, we can write

$$D_n\theta = \sum_{i_1=1}^{\beta_{\ell_1}} \cdots \sum_{i_s=1}^{\beta_{\ell_s}} a_{\tau_{\ell_1}^{i_1} \cdots \tau_{\ell_s}^{i_s}} D_n (1 + T_{\ell_1})^{i_1} \cdots (1 + T_{\ell_s})^{i_s}$$

where $\beta_{\ell_i} := \#\mathcal{G}_{\ell_i}$. Modulo p^k, Proposition A.4.2 implies that $D_n T_{\ell_i} = -N_{\ell_i}$ and $D_n T_{\ell_i}^2 = 0$. Thus

$$D_n\theta \equiv a_{\tau_{\ell_1}^{i_1} \cdots \tau_{\ell_s}^{i_s}} (1 - i_1 N_{\ell_1}) \cdots (1 - i_s N_{\ell_s}) \mod p^k$$

Since $D_n\theta$ is Galois invariant, the only non-vanishing summand in the right hand side is the multiple of $N_{\ell_1} \cdots N_{\ell_s}$. Since $i_i = \log_{\tau_{\ell_i}}(\sigma)$, we have

$$D_n\theta \equiv (-1)^{\nu(n)} \sum_{\sigma \in \mathcal{G}_n} a_\sigma \prod_{\ell | n} \log_{\tau_\ell}(\sigma) \mod p^k$$

□

Proposition 8.3.12 motivates definition 8.1.22 of the (twisted) Kurihara numbers.

Remark 8.3.13. The definition of $\delta_{n,\chi}$ depends on the choices of the generators $\eta_\ell \in (\mathbb{Z}/\ell)^\times$ for every $\ell \mid n$. However, by remark 8.3.11, the p-adic valuation of $\delta_{n,\chi}$ is well defined independently of the choices made in the construction of $\delta_{n,\chi}$.

From propositions 8.3.9 and 8.3.12, we obtain the following.

Corollary 8.3.14. For every primitive character χ of $\mathrm{Gal}(K/\mathbb{Q})$ and every $n \in \mathcal{N}_k$,

$$D_n e_\chi(\Theta_{cn}) \equiv \frac{\varphi(n)}{n}(1 - p^{-1}a_p\overline{\chi}(p) + p^{-1}\mathbf{1}_N(p)\overline{\chi}(p)^2)\delta_{n,\overline{\chi}}e_{\chi\times\mathbf{1}_n}$$

modulo $\frac{p^{k+k_{\overline{\chi}}}}{\varphi(c)}\mathbb{Z}_p[\chi][G_{\mathbb{Q}(\mu_c)/\mathbb{Q}} \times \mathcal{G}_n]$.

Proof. By Proposition 8.3.12,

$$D_n\chi(\theta_{cn}) \equiv \delta_{n,\overline{\chi}}N_n = \delta_{n,\overline{\chi}}\varphi(n)e_{\mathbf{1}_n} \mod p^k\mathbb{Z}_p[\mathcal{G}_n]$$

After multiplying both sides by the Euler product

$$\left(1 - p^{-1}a_p\overline{\chi}(p) + \mathbf{1}_N(p)p^{-1}\overline{\chi}(p)^2\right)$$

and by the idempotent element associated with the character χ, the congruence holds modulo $p^{k+k_{\overline{\chi}}}\varphi(c)^{-1}$. Thus, the corollary follows from Proposition 8.3.9. □

We can adapt Corollary 8.3.14 to describe the χ-part of $\Theta_{K(n)}$ in terms of the Kurihara numbers.

Corollary 8.3.15. For every primitive character χ of $\mathrm{Gal}(K/\mathbb{Q})$, we have that

$$D_n e_\chi(\Theta_{K(n)}) = \frac{\varphi(n)}{n}(1 - p^{-1}a_p\overline{\chi}(p) + p^{-1}\mathbf{1}_N(p)\overline{\chi}(p)^2)\delta_{n,\overline{\chi}}e_{\chi\times\mathbf{1}_n}$$

modulo $p^{k+k_{\overline{\chi}}}\mathbb{Z}_p[\chi][G \times \mathcal{G}_n]$.

Proof. It follows from Corollary 8.3.14, projecting the congruence to $\mathbb{Z}_p[\chi][G \times G_n]$.

Since both sides of that equation are invariant under the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_c)/K)$, whose order is $\varphi(c)/d$, we can remove the denominator $\varphi(c)$ from the ideal of the congruence. □

By equation (8.2.24), we obtain the following corollary.

Corollary 8.3.16. For every primitive character χ of $\mathrm{Gal}(K/\mathbb{Q})$ and every $n \in \mathcal{N}_k$, the Kolyvagin derivative $D_n(w_{\mathbb{Q}(n),\chi})$ can be computed, modulo $p^{k+k_\chi}\mathbb{Z}_p[G \times \mathcal{G}_n](\xi_{K(n)})$, by the expression

$$\frac{(-1)^{\nu(n)}\chi(n)}{n}(1 - p^{-1}a_p\chi(p) + p^{-1}\mathbf{1}_N(p)\chi(p)^2)\delta_{n,\chi}\varphi(c)e_{\overline{\chi}}(\zeta_c)$$

Proof. By the transitivity of the trace,

$$\mathrm{Tr}_{K(n)/K}(\xi_{K(n)}) = (-1)^{\nu(n)}\ \mathrm{Tr}_{\mathbb{Q}(\mu_c)/K}\left(\sum_{\widetilde{c}|d|c} \zeta_d^{(n^{-1})}\right)$$

Denote

$$\widetilde{\xi}_K = \mathrm{Tr}_{\mathbb{Q}(\mu_c)/K}\left(\sum_{\widetilde{c}|d|c} \zeta_d^{(n^{-1})}\right)$$

Since χ has conductor c, by Lemma 8.2.17

$$e_{\chi\times\mathbf{1}_n}(\xi_{K(n)}) = \frac{(-1)^{\nu(n)}}{\varphi(n)} e_\chi(\widetilde{\xi}_K) = \frac{(-1)^{\nu(n)}\overline{\chi}(n)}{\varphi(n)}[\mathbb{Q}(\mu_c) : K]e_\chi(\zeta_c)$$

Since $[\mathbb{Q}(\mu_c) : K] = \frac{\varphi(c)}{d}$, by equation (8.2.24) and Corollary 8.3.15, we obtain that

$$D_n(w_{\mathbb{Q}(n),\chi}) \equiv \frac{(-1)^{\nu(n)}\chi(n)}{n}(1-p^{-1}a_p\chi(p)+p^{-1}\chi(p)^2)\delta_{n,\chi}\varphi(c)e_{\overline{\chi}}(\zeta_c)$$

□

8.3.3 Kato's Kolyvagin system

In this section, we aim to use the expression Theorem 8.3.16 to compute the indices of the localization at p of Kato's Kolyvagin system, needed to apply Theorems 3.3.7 and 3.4.1.

Theorem 8.3.17. The index of the localisation of the classes in Kato's Kolyvagin system can be computed as

$$\mathrm{ind}(\mathrm{loc}_s^p(\kappa_{n,\chi}), H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T(\chi))) = \mathrm{ind}(\delta_{n,\chi}, \mathcal{O}_d)$$

Remark 8.3.18. Theorem 8.1.27 is an immediate consequence of Theorem 8.3.17.

In order to prove this theorem, we need the following analogue of Proposition 8.2.22, concerning the interaction of the dual exponential map with restrictions.

Proposition 8.3.19. Let $c \in H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, V \otimes \mathcal{O}_d(\chi))$ and denote its restriction by $d = \mathrm{res}_{\mathbb{Q}(n)/\mathbb{Q}}(c) \in H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}(n), V \otimes \mathcal{O}_d(\chi))$. Then

$$\exp^*_{\omega_E,\overline{\chi}}(c) = \exp^*_{\omega_E,\overline{\chi}}(d) \in (\mathbb{Q}(n) \otimes \mathbb{Q}_p \otimes \mathcal{O}_d(\chi))^{\mathcal{G}_n}$$

Proof. Localising at primes above p, we get that

$$\mathrm{loc}_p^{\mathrm{s}}(d) = \left(\bigoplus_{v|p} \mathrm{res}_{\mathbb{Q}(n)_v/\mathbb{Q}_p} \right) \left(\mathrm{loc}_p^{\mathrm{s}}(c) \right)$$

By [NSW00, Proposition 1.5.3 (iv)], if $\mathrm{loc}_p^{\mathrm{s}}(c)^\vee$ and $\mathrm{loc}_p^{\mathrm{s}}(d)^\vee$ are considered as maps in duals of the Bloch-Kato cohomology groups, then

$$\mathrm{loc}_p^{\mathrm{s}}(d)^\vee = \mathrm{loc}_p^{\mathrm{s}}(c)^\vee \circ \left(\bigoplus_{v|p} \mathrm{cor}_{\mathbb{Q}(n)_v/\mathbb{Q}_p} \right)$$

By [NSW00, Proposition 1.5.2], $\exp^*_{\omega_E,\overline{\chi}}(d) = \exp^*_{\omega_E,\overline{\chi}}(c) \circ N_{\mathbb{Q}(n)/\mathbb{Q}}$

By the identification in (8.5),

$$\exp^*_{\omega_E,\overline{\chi}}(c) = \exp^*_{\omega_E,\overline{\chi}}(d) \qquad \square$$

Proof of Theorem 8.3.17. In Corollary 8.3.16, we have computed the value of the dual exponential map $\exp^*_{\omega_E,\overline{\chi}}(D_n z_{\mathbb{Q}(n),\chi})$ modulo $p^{k+k_\chi}\mathbb{Z}_p[\mathcal{G}_n](\xi_{K(n)})$, which is given by the expression

$$\frac{(-1)^{\nu(n)}\chi(n)}{n}(1 - p^{-1}a_p\chi(p) + p^{-1}\chi(p)^2)\delta_{n,\chi}\varphi(c)e_{\overline{\chi}}(\zeta_c)$$

Since $p^{k+k_\chi}\mathbb{Z}_p[\mathcal{G}_n](\xi_{K(n)}) \cap K$ is contained in $p^{k+k_\psi}\mathcal{O}_p$, the congruence also holds modulo this ideal too.

The goal of this section is to relate this value with Kato's Kolyvagin system. After that, we will check that the p-divisibility of Kato's Kolyvagin system and the Kurihara numbers coincide.

As defined in Definition A.4.7, $\kappa_{n,\chi}$ is the preimage of $D_n z_{\mathbb{Q}(n),\chi}$ under the restriction map

$$\mathrm{res}_{\mathbb{Q}(n)/\mathbb{Q}} : \; H^1(\mathbb{Q}, T/p^k \otimes \mathcal{O}_d(\chi)) \to H^1(\mathbb{Q}(n), T/p^k \otimes \mathcal{O}_d(\chi))^{\mathcal{G}_n}$$

If we understand $\mathrm{loc}_p^{\mathrm{s}}(\kappa_{n,\chi})$ and $\mathrm{loc}_v^{\mathrm{s}}(D_n\omega_{\mathbb{Q}(n),\chi})$ as maps in the duals of $H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T^* \otimes \mathcal{O}_d(\overline{\chi}))$ and $H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}_p, T^* \otimes \mathcal{O}_d(\overline{\chi}))$, we obtain by [NSW00, Proposition 1.5.3(iv)] that

$$\mathrm{loc}_v^{\mathrm{s}}(D_n w_{n,\chi}) = \mathrm{loc}_p^{\mathrm{s}}(\kappa_{n,\chi}) \circ \mathrm{cor}_{\mathbb{Q}(n)/\mathbb{Q}}$$

Under the isomorphism in Corollary 8.2.27, the weak Kato's Kolyvagin system is

$$\widetilde{\kappa}_{n,\chi} = \frac{(-1)^{\nu(n)}\chi(n)}{n}(1 - p^{-1}a_p\chi(p) + p^{-1}\chi(p)^2)\delta_{n,\chi}\varphi(c)e_{\overline{\chi}}(\zeta_c)$$

In order to obtain Kato's Kolyvagin system κ_χ from $\widetilde{\kappa}_\chi$, we need to apply the construction in Definition A.4.9. However, in this case, $\kappa_\chi = \widetilde{\kappa}_\chi$. In fact, by the definition of Kolyvagin primes, $a_\ell = 2 \mod p^k$ for every $\ell \mid n$. Therefore, for every ℓ, we have that

$$P_\ell = X^2 - a_\ell X + \ell \equiv (X-1)^2 \mod p^k$$

Therefore, for every $\ell \mid n$, $\rho_\ell(P_\ell(\mathrm{Frob}_{\pi(\ell)}^{-1})) = 0$. Hence, in the formula of Definition A.4.9, the only term that does not vanish is the one associated with the trivial permutation, so $\kappa_{n,\chi} = \widetilde{\kappa}_{n,\chi}$.

Since p is unramified in $K/\mathbb{Q}$ by (K1), then $\varphi(c)e_{\overline{\chi}}(\zeta_c)$ generates the module $(\mathcal{O}_d)_{\overline{\chi}}$, and taking into account the description of the image of the dual exponential map in Theorem 8.2.25, we get that

$$\mathrm{ind}\big(\mathrm{loc}_s^p(\kappa_{n,\chi}), H^1_{/\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T(\chi))\big) = \mathrm{ind}\big(\delta_{n,\chi}, (\mathcal{O}_d)_{\overline{\chi}}\big)$$

□

8.3.4 Primitivity of Kato's Euler system

In this section, we prove that the Kolyvagin systems obtained from Kato's Euler system are primitive under our assumptions. This is a consequence of the Iwasawa main conjecture and Theorem 6.5.2. This question have been already studied in the literature, under different assumptions. See, for example, [Gri05] or [Kim25].

Kato's Kolyvagin system over the cyclotomic $\mathbb{Z}_p$-extension $\kappa^{(\infty)} \in \mathbf{KS}(T(\chi) \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$ and its analogue over the rational numbers $\kappa \in \mathbf{KS}(T(\chi), \mathcal{F}^{\mathrm{can}})$ are obtained from Kato's Euler system $\mathbf{z}$ as

$$\kappa^{(\infty)} = \psi_\infty(\mathbf{z}), \quad \kappa = \psi(\mathbf{z})$$

Note that, Remark A.4.18 implies that the primitiviy of $\kappa^{(\infty)}$ is equivalent to the full Iwasawa main conjecture for the twisted modular form f_χ.

When we can only guarantee the Iwasawa main conjecture localised at the ideal (X), there is a primitive Kolyvagin system $\kappa^{(\infty)} \in \mathbf{KS}(T(\chi) \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$ and an element $f \in \Lambda \setminus X\Lambda$ such that $\kappa^{(\infty)} = f\widetilde{\kappa^{(\infty)}}$.

By Assumptions (K3) and (K4), $T(\chi) \otimes \Lambda$ satisfies Assumption (Tam). Indeed, $\Sigma_{\mathcal{F}^{\mathrm{can}}}$ is formed by the primes dividing $Ncp\infty$. For the primes $\ell \mid p\infty$, Assumptions (E0) and (K3) guarantees that

$H^0(\mathbb{Q}_\ell, E[p^\infty] \otimes \Lambda) = 0$. When $\ell \mid c$, then ℓ is a good reduction prime, so

$$H^0(K^{\mathrm{ur}}/K, E[p^\infty] \otimes \mathcal{O}_d(\overline{\chi})) = E[p\infty]$$

is a p-divisible group. since $K^{\mathrm{ur}}/\mathbb{Q}_\infty$ has profinite order prime to p by Assumption (K2), then $H^0(K^{\mathrm{ur}}/K, E[p^\infty] \otimes \mathcal{O}_d(\overline{\chi}))$ is also a p-divisible group. When $\ell \mid N$, Assumption (K4) implies that $H^0(K^{\mathrm{ur}}/K, E[p^\infty])$ is also p-divisible. The same argument proves the divisibility of $H^0(K^{\mathrm{ur}}/K, E[p^\infty] \otimes \mathcal{O}_d(\overline{\chi}))$ in this case. Then Assumption (Tam) follows from 6.1.14.

Therefore, Theorem 6.5.2 implies that the projection map

$$\Pi : \ \mathbf{KS}(T(\chi) \otimes \Lambda, \mathcal{F}^{\mathrm{can}}) \twoheadrightarrow \mathbf{KS}(T(\chi), \mathcal{F}^{\mathrm{can}})$$

is surjective and has kernel $X\mathbf{KS}(T \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$. Since $f \notin X\Lambda$, then

$$\kappa = \Pi\big(\kappa^{(\infty)}\big) \neq 0$$

Hence, there is a primitive Kolyvagin system $\widetilde{\kappa} \in \mathbf{KS}(T(\chi), \mathcal{F}^{\mathrm{can}})$ anda non-zero $\alpha \in \mathcal{O}_d$ such that $\kappa = \alpha\widetilde{\kappa}$. Note that Theorem 6.5.2 ensures that $\alpha \in \mathcal{O}_d^\times$ when the full Iwasawa main conjecture holds for f_χ.

By Theorem 5.4.10, the p-adic valuation of α can be computed using the theta ideals at infinity.

Proposition 8.3.20. As submodules of $\mathbf{KS}(T, \mathcal{F}^{\mathrm{can}})$, the following equality holds:

$$\kappa\mathbf{KS}(T, \mathcal{F}^{\mathrm{can}}) = \boldsymbol{\Theta}_\infty\big(\widetilde{\delta_\chi}\big)\mathbf{KS}(T, \mathcal{F}^{\mathrm{can}})$$

Proof. Note that, Proposition 8.1.5 implies that $\chi(\mathcal{F}_{\mathrm{BK}}) = 0$ and $\chi(\mathcal{F}^{\mathrm{can}}) = 1$. By Theorem 5.4.10, we have that

$$\kappa\mathbf{KS}(T, \mathcal{F}^{\mathrm{can}}) = \boldsymbol{\Theta}_\infty^{(0)}\big(\kappa, \mathcal{F}_{\mathrm{BK}}\big)\mathbf{KS}(T, \mathcal{F}^{\mathrm{can}})$$

Then the proposition follows from Theorem 8.3.17 □

Since Kato's Kolyvagin system is non zero, we can deduce the following corollary.

Corollary 8.3.21. The analytic ideal $\boldsymbol{\Theta}_\infty\big(\widetilde{\delta_\chi}\big)$ is non-zero.

In addition, the ideal $\Theta_\infty(\widetilde{\delta_\chi})$ can be seen as a numerical check of the Iwasawa main conjecture.

Proof of Theorem 8.1.37. If the Iwasawa main conjecture holds, κ is primitive by Theorem 6.5.5.

Conversely, assume that κ is primitive, which forces $\kappa^{(\infty)}$ to be primitive. Hence, the Iwasawa main conjecture for f_χ holds by Remark A.4.18 (see also Remark 8.1.17). $\square$

8.4 Structure of the Selmer group

The goal of this section is to conclude the proofs of Theorems 8.1.28 and 8.1.32. This will be achieved as an application of Theorems 3.3.7 and 3.4.1, depending on whether χ is a quadratic character or not.

Recall that we constructed in 8.2.20 an Euler system for the Galois representation $T(\chi)$. Then the Kolyvagin derivative process in Corollary A.4.11 constructs a Kolyvagin system $\kappa_\chi \in \mathrm{KS}(\mathcal{F}^{\mathrm{can}})$. Note that $\mathcal{F}_{\mathrm{BK}} \leq \mathcal{F}^{\mathrm{can}}$ and, by Proposition 8.1.5, their core ranks are $\chi(\mathcal{F}_{\mathrm{BK}}) = 0$ and $\chi(\mathcal{F}^{\mathrm{can}}) = 1$. Taking into account Proposition 8.1.4, we are in the situation of Theorem 5.4.5 and, when χ is not a quadratic character, under the hypothesis of Theorem 5.4.6.

We want to prove Theorem 8.1.28 as an application of Theorem 3.3.7. However, the conclusion of Theorem 8.1.28 is stronger than the one in Theorem 3.3.7. This can be achieved using a functional equation satisfied by the analytic theta ideals.

Recall that N is the conductor of the elliptic curve. The functional equation of the analytic theta ideals can be deduced from the functional equation satisfied by the Mazur-Tate elements.

Proposition 8.4.1. ([MT87, §1.6]) Let

$$\iota :\ \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})] \to \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$$

be the map induced from the involution in $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ that sends every element to its inverse. Then

$$\iota(\theta_n) = \varepsilon\sigma_{-N}\theta_n$$

where $\varepsilon \in \{\pm 1\}$ is the root number of the elliptic curve and $\sigma_N \in \mathrm{Gal}(\mathbb{Q}(\mu_n/\mathbb{Q}))$ is the Galois automorphism that sends ζ_n to ζ_n^N.

Considering character parts, we obtain the following result.

Corollary 8.4.2. If χ is a character of G, then

$$\iota\left(\overline{\chi}(\theta_{K(n)})\right) = \varepsilon\chi(-N)\chi(\theta_{K(n)}) \in \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$$

Corollary 8.4.3. Assume χ is a quadratic character. Depending on the root number ε of the elliptic curve and the value $\chi(-N)$, we have that

- If $\varepsilon\chi(-N) = 1$, then $\delta_{n,\chi} = 0$ when n has an odd number of prime divisors.
- If $\varepsilon\chi(-N) = -1$, then $\delta_{n,\chi} = 0$ when n has an even number of prime divisors.

Proof. Assume $\ell \in \mathcal{P}(T(\chi)/p^k)$. Then the construction of the Kolyvagin derivative implies for every $\theta \in \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})]$ that

$$D_\ell \iota(\theta) \equiv -D_\ell(\theta) \mod p^k$$

Therefore, for every $n \in \mathcal{N}_k$,

$$D_n\overline{\psi}(\theta_{K(n)}) = (-1)^{\nu(n)}\varepsilon\psi(N)D_n\psi(\theta_{K(n)})$$

Since $\chi = \overline{\chi}$, we have that

$$\delta_{n,\chi}(1 - (-1)^{\nu(n)}\varepsilon\chi(-N)) = 0$$

Since $\chi(-N) = \pm 1$, we have two possible cases. ☐

The previous result can be restated in terms of the theta ideals.

Corollary 8.4.4. Assume χ is a quadratic character. Then

- If $\varepsilon\chi(-N) = 1$, then $\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = 0$ when i is odd.
- If $\varepsilon\chi(-N) = -1$, then $\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = 0$ when i is even.

Now we can conclude the proof of Theorem 8.1.28.

Proof of Theorem 8.1.28. Note that, by remark 8.1.34, we can assume χ is primitive.

By Remark 8.1.7, the Bloch-Kato and canonical Selmer structures satisfy that $\chi(\mathcal{F}_{\mathrm{BK}}) = 0$ and $\chi(\mathcal{F}^{\mathrm{can}}) = 1$. By Theorem 8.1.27, we have that

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \Theta(\kappa_{\mathrm{Kato}}^{(k)}, \mathcal{F}_{\mathrm{BK}})$$

Assume first that $\chi \neq \overline{\chi}$. In this case, $T(\chi)/p^k$ satisfies Assumption (NSD). By Theorems 3.3.15 and 8.1.27, there exists a primitive Kolyvagin system $\widetilde{\kappa} \in \mathrm{KS}(\mathcal{F}^{\mathrm{can}})$ such that

$$\kappa = \Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big)\widetilde{\kappa}$$

Theorem 3.4.1 now implies that

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \Theta_i(\kappa_{\mathrm{Kato}}^{(k)}, \mathcal{F}_{\mathrm{BK}}) = \Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big)\mathrm{Fitt}^i_{\mathcal{O}_d}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(\mathbb{Q}, T(\chi)/p^k)\big)$$

Assume now that $\chi = \overline{\chi}$. Then $T(\chi)/p^k$ does not satisfy Assumption (NSD), so only Theorem 3.3.7 can be applied. However, Corollary 8.4.3 implies that $\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big)$ can be non-zero only when i has the same parity as $\varepsilon\chi(-N)$, which coincides with the parity of $r\chi(-N)$. Then the second condition of Theorem 3.3.7 implies that

$$\Theta_i\big(\widetilde{\delta}_\chi^{(k)}\big) = \mathrm{Fitt}^i_{\mathcal{O}_d/p^k}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K, E[p^k])\big)\Theta_\infty\big(\widetilde{\delta}_\chi^{(k)}\big) \text{ if } (-1)^{i-\varepsilon\chi(-N)} = 0$$

□

Proof of Theorem 8.1.28. By Theorem 8.1.28 and taking limits, we have, whenever $\chi \neq \overline{\chi}$ or $\chi = \overline{\chi}$ and $(-1)^i = (-1)^{i-\varepsilon\chi(-N)}$, that

$$\mathrm{Fitt}^i_{\mathcal{O}_d}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K, E[p^\infty])^\vee\big) = (p^{n_i})$$

In addition, when $\chi = \overline{\chi}$ and $(-1)^i \neq (-1)^{\varepsilon\chi(-N)}$, Corollary 5.4.8 implies that

$$\mathrm{Fitt}^i_{\mathcal{O}_d}\big(H^1_{\mathcal{F}_{\mathrm{BK}}}(K, E[p^\infty])^\vee\big) = \big(p^{\frac{n_{i-1}-n_{i+1}}{2}}\big)$$

Then Theorem 8.1.28 follows from the structure theorem and Proposition B.2.6 (see also Remark B.2.8) . □

8.5 Examples

We end this article by showing some examples of computations o the Selmer group of elliptic curves. All the computations are done using Sagemath [The]. For all the elliptic curves E and abelian extensions $K/\mathbb{Q}$ appearing in these examples, we will assume that $Ш(E/K)$ is finite.

Example 8.5.1. Consider the elliptic curve 196 794cd1 in Cremona's database and the prime $p = 5$. Over the abelian extension $K = \mathbb{Q}(\mu_7)$, we can determine the full group structure of the Selmer group. We proceed by studying the twisted Kurihara number for every Dirichlet character of conductor dividing 7.

When χ is the trivial character, we compute $\delta_{1,\chi} = 0$, so the rank of the Mordell-Weil group $E(\mathbb{Q})$ is at least one. To obtain further information about the structure of the Selmer group, we need to compute $\Theta_{1,\chi}$. The smallest Kolyvagin prime $\ell = 93\,251$ satisfies that $\mathrm{ord}_5(\delta_{\ell,\chi}) = 2$. It guarantees that $\mathrm{rank}(E(\mathbb{Q})) = 1$ and

$\#Ш(E/\mathbb{Q})[5^\infty] \mid 25$. If we compute $\mathrm{ord}_5(\delta_{\ell,\chi})$ for the smallest Kolyvagin primes, we will obtain the value 2 for approximately the 80% of the computations and a higher value in the remaining cases. That would lead us to guess that $\Theta_2 = 25\mathbb{Z}_5$, which is equivalent to $\#Ш(E/\mathbb{Q})[5^\infty] = 25$. However, we cannot prove it with only a finite amount of computations.

But we can use a different method to compute the order of the Tate-Shafarevich group using the Iwasawa main conjecture, which can be numerically verified using Theorem 8.1.37. The smallest Kolyvagin primes for $k = 1$ are 11, 31 and 131. Their product $n = 44\,671$ satisfies that $\delta_{n,\chi} \in \mathbb{Z}_5^\times$, so the Iwasawa main conjecture holds true in this elliptic curve.

Using Sagemath, we can check that the 5-adic L-function can be written as

$$\mathrm{char}(X_\infty) = \mathcal{L}_5(E,T) = (5^2 + O(5^3))T + O(T^2)$$

Under our assumptions, Mazur's control theorem works perfectly, so we can conclude that $\mathrm{Sel}(\mathbb{Q}, T_5E) \cong \mathbb{Z}_5 \times \mathbb{Z}_5/(5) \times \mathbb{Z}_5/(5)$. Therefore,

$$Ш(E/\mathbb{Q})[5^\infty] = Ш(E/K)[5^\infty]^{\mathrm{Gal}(K/\mathbb{Q})} \cong \mathbb{Z}_5/(5) \times \mathbb{Z}_5/(5)$$

If χ is the quadratic character of conductor 7, then $\delta_{1,\chi} = 0$, so $\mathrm{rank}(E(K)_\chi) \geq 1$. In order to determine the full structure of the twisted Selmer group, we need to compute $\Theta_{1,\chi}$. Since 11 is a Kolyvagin prime and $\mathrm{ord}_5(\delta_{11,\chi}) = 0$, we deduce that $\Theta_1 = \mathbb{Z}_5$. Hence $\mathrm{rank}(E(K)_\chi) = 1$ and $Ш(E/K)[5^\infty]_\chi = \{0\}$.

For all other characters of conductor 7, we compute $\mathrm{ord}_5(\delta_{1,\chi}) = 1$, which implies that $Ш(E/K)_\chi \cong \mathcal{O}_6/(5)$.

All the information above is enough to compute the structure of the Selmer group $\mathrm{Sel}(K, T_5E) \otimes \mathcal{O}_6$ as an $\mathcal{O}_6[\mathrm{Gal}(K/\mathbb{Q})]$-module. By Proposition 8.1.8, the Fitting ideal are then given by the expressions

$$\begin{aligned}
\mathrm{Fitt}^0_{\mathbb{Z}_5}(\mathrm{Sel}(K, T_5E)) &= \frac{5}{3}(2\sigma_1 - \sigma_2 - \sigma_4) \\
\mathrm{Fitt}^1_{\mathbb{Z}_5}(\mathrm{Sel}(K, T_5E)) &= 5\sigma_1 + 4(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6) \\
\mathrm{Fitt}^2_{\mathbb{Z}_5}(\mathrm{Sel}(K, T_5E)) &= \frac{5}{3}\sigma_1 + \frac{2}{3}(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6)
\end{aligned}$$

By Proposition 8.1.8, the Selmer group $\mathrm{Sel}(K, T_5E)$ is isomorphic, as $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$-modules. to

$$\frac{\mathbb{Z}_p[G]}{(5(2\sigma_1 - \sigma_2 - \sigma_4))} \times \left(\frac{\mathbb{Z}_p[G]}{(5\sigma_1 + 2(\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6))}\right)^2$$

Example 8.5.2. Let E be the elliptic curve 35a1 in Cremona's database and let $p = 7$. For a Dirichlet character χ of conductor 51 and order 8 such that $\chi(35) = -1$ and $\chi(37)$ is a primitive 8^{th} root of unity ζ_8 satisfying that $\zeta_8^2 + 3\zeta_8 + 1 \in 7\mathbb{Z}_7$.

We can compute $\operatorname{ord}_7(\delta_{1,\chi}) = 2$, so Theorem 8.1.28 implies that $\operatorname{rank}(E(\mathbb{Q}(\mu_{51})))_\chi = 0$ and

$$\#Ш(E/\mathbb{Q}(\mu_{51}))[7^\infty]_\chi = \big(\#(\mathcal{O}_{16}/(7))\big)^2$$

Note that χ is not self-dual, so there are no non-degenerate pairings defined on the twisted Selmer group that determine its structure. In this case, the computation of $\Theta_{1,\chi}$ is what determines whether the Tate-Shafarevich group is isomorphic to $\mathcal{O}_{16}/(7^2)$ or $\mathcal{O}_{16}/(7) \times \mathcal{O}_{16}/(7)$.

The smallest Kolyvagin prime is $\ell = 2\,801$ and satisfies that $\delta_{\ell,\chi} \in \mathcal{O}_8^\times$. Hence $\Theta_{1,\chi} = \mathcal{O}_8$ and

$$\#Ш(E/\mathbb{Q}(\mu_{51}))[7^\infty]_\chi \cong \mathcal{O}_{16}/(7^2)$$

Example 8.5.3. Consider the elliptic curve 11a1 in Cremona's database, the abelian extension $K = \mathbb{Q}(\mu_{61})$ and the prime $p = 101$.

Let χ be the character of conductor 61 and order 20 such that $\chi(2)$ is the unique primitive 20^{th} root of unity in $60 + 101\mathbb{Z}_{101}$. Then

$$\operatorname{ord}_{101}(\delta_{1,\chi}) = \operatorname{ord}_{101}(\delta_{1,\overline{\chi}}) = 1$$

Theorem 8.1.32 then implies that

$$\operatorname{Sel}(\mathbb{Q}, T_pE \otimes \mathbb{Z}_{101}(\chi)) \cong \operatorname{Sel}(\mathbb{Q}, T_pE \otimes \mathbb{Z}_{101}(\overline{\chi})) \cong \mathbb{Z}_{101}/(101)$$

For the quadratic character χ' of conductor 61, we obtain that $\delta_{1,\chi'} = 0$, so Theorem 8.1.32 implies that $\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{Sel}(\mathbb{Q}, T_pE\otimes)) \geq 1$. To determine the full structure of this Selmer group, we need to compute $\Theta_{1,\chi'}$. The smallest Kolyvagin prime is $\ell' = 64\,237$ and satisfies that $\operatorname{ord}_{101}(\delta_{\ell',\chi}) = 0$. Hence, $\Theta_{i,\chi} = \mathbb{Z}_p$, so

$$\operatorname{Sel}(\mathbb{Q}, T_pE \otimes \mathbb{Z}_{101}) \cong \mathbb{Z}_{101}$$

Finally, consider a character χ'' of conductor 61 and order 6. It satisfies that $\chi(2)$ is a 6^{th}-root of unity. Then $\delta_{1,\chi''} = \delta_{1,\overline{\chi''}} = 0$. The smallest Kolyvagin prime for these characters is $\ell'' = 2\,528\,233$, satisfying that

$$\operatorname{ord}_{101}(\delta_{\ell'',\chi''}) = \operatorname{ord}_{101}(\delta_{\ell'',\overline{\chi''}}) = 0$$

Hence,

$$\mathrm{Sel}\big(\mathbb{Q}, T_pE \otimes \mathcal{O}_6(\chi'')\big) \cong \mathrm{Sel}\big(\mathbb{Q}, T_pE \otimes \mathcal{O}_6(\overline{\chi''})\big) \cong \mathcal{O}_6$$

Overall, assuming the finiteness of the Tate-Shafarevich group, we know that $\mathrm{rank}(E(K)) = 3$. Furthermore, we know how the rank grows along the subextensions of $K/\mathbb{Q}$. Indeed, if K_2 and K_3 are the subextensions of degrees 2 and 3, respectively, we know that $\mathrm{rank}(E(K_2)) = 1$ and $\mathrm{rank}(E(K_3)) = 2$.

Similarly, if L is a subextension of $K/\mathbb{Q}$, then $Ш(E/L)$ would have order 101^2 if $[L : \mathbb{Q}] \in 20\mathbb{Z}$ and would be trivial otherwise.

Example 8.5.4. We can use this method to find Tate-Shafarevich groups divisible by large primes. Consider the elliptic curve 27a1 in the Cremona tables and the prime $p = 472\,558\,791\,937$. Let χ be the Dirichlet character of conductor 89 satisfying that $\chi(3)$ is the unique primitive 88^{th} root of unity in $382\,613\,086\,515 + p\mathbb{Z}_p$. Then

$$\mathrm{ord}_p(\delta_{1,\chi}) = 1$$

Therefore, we can conclude that

$$\mathrm{Sel}(\mathbb{Q}, T_pE \otimes \mathbb{Z}_p(\chi)) \cong \mathbb{Z}_p/(p)$$

Appendix A

From Euler systems to Kolyvagin systems

In the previous chapter, we have described how to compute the Fitting ideals of Selmer groups using Kolyvagin systems but we have said nothing about where Kolyvagin systems come from or how to construct them. The short answer is that they can be obtained from Euler systems.

From an Euler system, one can apply a Kolyvagin derivative operator in order to construct a Kolyvagin system for the canonical Selmer structure, defined in Definition 2.3.7.

We will describe here the construction of Kolyvagin systems when the core rank of the canonical Selmer structure is one. This construction has been generalised in [BSS25] to Galois representations in which the core rank of the canonical Selmer structure is higher.

A.1 Euler systems

Roughly speaking, an Euler system is a collection of elements in the different Galois cohomology groups $H^1(F,T)$, where F runs through the abelian extensions of K, which satisfy norm-compatibility relations involving Euler factors. Before defining Euler systems precisely, we need to introduce the concept of Euler factors associated with a Galois representation, which are also used to define the L-functions.

Definition A.1.1. Let T be a p-adic Galois representation and let ℓ be a prime of K not dividing p. The *Euler factor* of T at ℓ is defined as the polynomial

$$P_\ell(x) = \det\bigl(1 - \mathrm{Frob}_\ell^{-1} x | H^0(\mathcal{I}_\ell, T)\bigr) \in R[x]$$

where Frob_ℓ denoted the arithmetic Frobenius at ℓ.

We are now ready to give the definition of an Euler system.

Definition A.1.2. Let $\mathcal{K}/K$ be an abelian extension satisfying that

- $\mathcal{K}$ contains $K(q)$ for all but finitely many q.
- $\mathcal{K}/K$ contains a $\mathbb{Z}_p$-extension along which no finite prime splits completely.

If we denote by Ω the set of finite subextensions of $\mathcal{K}/K$, an *Euler system* is a collection of classes

$$\mathbf{c} = \bigl\{c_F \in H^1(F, T) : \ F \in \Omega\bigr\}$$

satisfying the following norm-compatibility relations: when $F \subset L \in \Omega$, then

$$\mathrm{cor}_{L/F}(c_L) = \left(\prod_{\ell \in \Sigma_{L/F}} P_\ell(\mathrm{Frob}_\ell^{-1})\right) c_F$$

where $\Sigma_{L/F}$ is the set of unramified primes, not dividing $p\infty$, of K which ramify in L/K but not in F/K.

Notation A.1.3. We denote the module of Euler systems by $\mathrm{ES}(T)$.

Notation A.1.4. For every Euler system $\mathbf{c}$, we denote by c_n the class $c_{K(n)}$, where $K(n)$ was defined as the maximal p-extension inside the ray class field modulo n.

As mentioned above, Euler factors are also constructed to define the L-function of the Galois representation. In fact, L-functions are obtained as the infinite product of all the Euler factors. However, note that Definition A.1.1 only defines these Euler factors for primes not dividing p. The extension of this definition to primes above p requires the use of Fontaine's crystalline period ring B_{crys}, which is a $\mathbb{Q}_p$-vector space endowed with a Frobenius automorphism ϕ.

Definition A.1.5. Let T be a p-adic Galois representation and let ℓ be a place of K dividing p. The Euler factor at ℓ is defined as

$$P_\ell(x) = \det\bigl(1 - \phi x | (T \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_{K_\ell}}\bigr)$$

Definition A.1.6. Let T be a p-adic Galois representation. Its *L-function* is defined as the complex function defined by infinite product

$$L(T,s) := \prod_{\ell\in\mathbb{P}_K} P_\ell(N_{K/\mathbb{Q}}(\ell)^{-s})^{-1} \text{ for } \mathrm{Re}(s) \gg 0$$

Remark A.1.7. It can be shown that there exists a constant C such that infinite product of Euler factors converges when $\mathrm{Re}(s) \geq C$. It is conjectured that when T is the Galois representation coming from a motive, the infinite product can be extended to a meromorphic complex function, which is also referred to as the L-function.

Remark A.1.8. Euler systems as in Definition A.1.2 can be used to construct Kolyvagin systems of rank r. Following the work of [Rub96] and [Per98], there is a more general definition of higher rank Euler systems, in which the classes live in exterior powers of the Selmer group, from which one can obtain Kolyvagin systems when the core rank of the canonical Selmer structure is positive.

A.2 Iwasawa main conjecture

There are different equivalent formulations of the Iwasawa main conjecture in the literature. We will use the one formulated by K. Kato in [Kat04]. This formulation does not contain p-adic L-functions but relates the characteristic ideal of a Selmer group with a zeta element, defined in loc.cit., which can be understood as the class of the Euler system over a $\mathbb{Z}_p$-extension of K.

Definition A.2.1. Let K_∞/K be a $\mathbb{Z}_p$-extension and denote by K_n/K the unique subextension of degree p^n. The Iwasawa cohomology is defined as

$$H^i_{\mathrm{IW}}(K,T) = \varprojlim_n H^i(k_n,T)$$

where the transition maps are correstrictions.

Definition A.2.2. Let $\mathbf{c} \in \mathrm{ES}(T)$ be an Euler system and let $K_\infty \subset \mathcal{K}$ be a $\mathbb{Z}_p$-extension. We define the Euler system element at K_∞ as

$$c_{K_\infty} = (c_{K_n})_{n\in\mathbb{N}} \in H^1_{\mathrm{IW}}(K,T)$$

We can now describe the Iwasawa main conjecture in the sense of Kato.

Definition A.2.3. Suppose that the canonical Selmer structure has core rank $\chi(\mathcal{F}^{\mathrm{can}}) = 1$. We say that an Euler system $\mathbf{c} \in \mathrm{ES}(T)$ satisfies the Iwasawa main conjecture if

$$\frac{H^1_{\mathrm{IW}}(K,T)}{\Lambda c_{\mathbb{Q}_\infty}}$$

is a torsion Λ-module and

$$\mathrm{Fitt}^0_\Lambda\left(\frac{H^1_{\mathrm{IW}}(K,T)}{\Lambda c_{\mathbb{Q}_\infty}}\right) \approx \mathrm{Fitt}^0_\Lambda(H^1_{(\mathcal{F}_{\mathrm{BK}})_p}(K,T^*)^\vee)$$

Remark A.2.4. The pseudo-equality between Fitting ideals appearing in Definition A.2.3 is equivalent to the equality of their characteristic ideals.

Remark A.2.5. When the core rank $\chi(\mathcal{F}^{\mathrm{can}}) > 1$, one can adapt Definition A.2.3 using higher rank Euler systems.

Remark A.2.6. When the localisation at p of the Euler system is related to a p-adic L-function via a big logarithm constructed by Perrin-Riou in [Per95], Definition A.2.3 is expected to have an analogue relating the characteristic ideal of the Bloch-Kato Selmer group and the p-adic L-function.

There is a weaker version of the Iwasawa main conjecture, which only concerns about one height prime ideal β.

Definition A.2.7. Assume that $\chi(\mathcal{F}^{\mathrm{can}}) = 1$ and let β be a height one ideal of the Iwasawa algebra. We say that an Euler system $\mathbf{c} \in \mathrm{ES}(T)$ satisfies the Iwasawa main conjecture localised at β if

$$\frac{H^1_{\mathrm{IW}}(K,T)}{\Lambda c_{\mathbb{Q}_\infty}}$$

is a torsion Λ-module and

$$\mathrm{Fitt}^0_{\Lambda_\beta}\left(\frac{H^1_{\mathrm{IW}}(K,T)_\beta}{\Lambda_\beta c_{\mathbb{Q}_\infty}}\right) \approx \mathrm{Fitt}^0_\Lambda(H^1_{(\mathcal{F}_{\mathrm{BK}})_p}(K,T^*)^\vee)_\beta$$

Remark A.2.8. The formula in Definition A.2.7 is equivalent to

$$\mathrm{ord}_\beta\left(\mathrm{char}\left(\frac{H^1_{\mathrm{IW}}(K,T)_\beta}{\Lambda_\beta c_{\mathbb{Q}_\infty}}\right)\right) = \mathrm{ord}_\beta\left(\mathrm{char}\left(H^1_{\mathcal{F}_{\mathrm{BK}_p}}(K,T^*)^\vee\right)\right)$$

Remark A.2.9. The full Iwasawa main conjecture is equivalent to the localised Iwasawa main conjecture localised at all height one prime ideals of Λ.

A.3 Examples of Euler system

A.3.1 Cyclotomic units

The cyclotomic units is an Euler system constructed for the Galois representation

$$T = \mathbb{Z}_p(1) = \varprojlim_{n\in\mathbb{N}} \mu_{p^n}$$

For every abelian extension $F/\mathbb{Q}$, the Kummer map induces an isomorphism

$$H^1(K,T) = \varprojlim_{n\in\mathbb{N}} F^\times/(F^\times)^{p^n} = F^\times \mathbin{\widehat{\otimes}} \mathbb{Z}_p$$

Fix a collection $\{\zeta_n : \ n \in \mathbb{Z}^+\}$, where ζ_n is a primitive n^{th}-root of unity in $\overline{\mathbb{Q}}$, such that

$$\zeta_{mn}^n = \zeta_n \ \forall n, m \in \mathbb{N}$$

A direct computation shows the following identity for a natural number $n > 1$ and a prime ℓ:

$$N_{\mathbb{Q}(\mu_{n\ell})/\mathbb{Q}(\mu_n)}(\zeta_{n\ell} - 1) = \begin{cases} (\zeta_n - 1) & \text{if } \ell \mid n \\ (\zeta_n - 1)^{1-\mathrm{Frob}_\ell^{-1}} & \text{if } \ell \mid n \end{cases}$$

In addition, we have, for every prime ℓ, the relation

$$N_{\mathbb{Q}(\mu_\ell)/\mathbb{Q}}(\zeta_\ell - 1) = (-1)^{\ell-1}\ell$$

This relation motivates the definition of the Euler system of cyclotomic units.

Definition A.3.1. The Euler system of cyclotomic units is determined by the values

$$\begin{aligned} c_{\mathbb{Q}(\mu_n)} &= N_{\mathbb{Q}(\mu_n)/\mathbb{Q}(\mu_n)}(\zeta_{np} - 1) \in \mathbb{Q}(\mu_m)^\times \hookrightarrow H^1(\mathbb{Q}(\mu_n), \mathbb{Z}_p(1)) \\ c_{\mathbb{Q}(\mu_n)^+} &= N_{\mathbb{Q}(\mu_n)/\mathbb{Q}} c_{\mathbb{Q}(\mu_n)} \in \left(\mathbb{Q}(\mu_m)^+\right)^\times \hookrightarrow H^1(\mathbb{Q}(\mu_n)^+, \mathbb{Z}_p(1)) \end{aligned}$$

where $\mathbb{Q}(\mu_n)^+$ is the maximal real subextension of $\mathbb{Q}(\mu_n)$.

It is easy to check that the cyclotomic units satisfy the definition of Euler systems since, for every $\ell \neq p$,

$$P_\ell(x) = \det(1 - \mathrm{Frob}_\ell^{-1} x|\mathbb{Z}_p(1)^*) = \det(1 - \mathrm{Frob}_\ell^{-1} x|\mathbb{Z}_p) = 1 - x$$

The Euler system can be twisted (see [Rub00, Proposition 2.4.2]) to obtain an Euler system for the representation

$$T_\chi = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} Z_p[\chi](\chi)$$

where $Z_p[\chi]$ is obtained by adjoining the values of χ by $\mathbb{Z}_p$ and the Galois action of an element $\sigma \in G_{\mathbb{Q}}$ on $Z_p[\chi](\chi)$ is the multiplication by $\chi(\sigma)$. The Bloch-Kato Selmer group in the dual Galois representation T_χ^*, which can be studied from the Kolyvagin system obtained from the cyclotomic units, is isomorphic the χ part of the class group of the fixed field of χ.

When χ is an odd character, the twisted Euler system vanishes, so cannot be used to bound the Selmer group. However, whenever χ is an even, non-trivial character and $p >$, the twisted cyclotomic units satisfy the Iwasawa main conjecture (see [Rub00, Theorem 3.2.10]).

A.3.2 Kato's Euler system

Kato's Euler system is a collection of zeta elements constructed in [Kat04] for the Galois representation associated with a modular form. More details about Kato's Euler system are given in §8.2.

It is interesting to first consider Kato's Euler system for the Tate module of an elliptic curve, since the existence of Néron periods produces a canonical choice of Kato's Euler system. It will be used in Chapter 8 to provide an explicit computation of the Galois structure of the classical Selmer group in terms of the modular symbols of the curve.

Theorem A.3.2. ([Kat21, Theorem 6.1]) Let $E/\mathbb{Q}$ be an elliptic curve defined over the rationals. Assume that $E[p]$ is an irreducible Galois representation. Then there exists an Euler system $\mathbf{z} = (z_F)_{F\in\Omega} \in \mathrm{ES}(T_pE)$ satisfying, for every Dirichlet character χ with conductor c and fixed field F, the interpolation relation

$$\sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})} \sigma\big(\exp^*_{\omega_E}(\mathrm{loc}_p(z_F))\big)\psi(\sigma) = \frac{L_{cp}(E,\psi,1)}{\Omega_E^{\psi(-1)}}$$

where $\Omega_E^{\chi(-1)}$ is one the Néron period associated with the curve, depending on whether χ is an even or an odd character, and $\exp^*_{\omega_E}$ is the dual exponential map associated with the Néron differential ω_E (see §8.2.1 for more details). Here, $L_{cp}(E,\psi,1)$ denotes the special value at $s = 1$ of the L-function $L_{cp}(E,\chi)$ which is defined by the infinite product in Definition A.1.6, for the twisted representation

$T_pE \otimes \mathcal{O}_d(\chi)$ (see §8.1.2), without including the Euler factors at primes dividing cp.

We can generalise this construction for Galois representations associated with weight 2 modular forms.

Notation A.3.3. Let f be a modular form of weight 2 and level N, let $Y_1(N)$ be the modular curve of associated with the group $\Gamma_1(N)$ and let K_f denote the field of coefficients of f. Fix a prime λ of K_f above f. Following [Kat04, §6.3], let $V_{K_{f,\lambda}}(f)$ be the Galois representation associated with f, defined as the maximal Hecke eigenquotient of $H^1_{\text{ét}}(Y_1(N), K_{g,\lambda})$ associated with f. In addition, denote $V_{\mathbb{C}}(f) = V_{K_g,\lambda}(f) \otimes_{K_{f,\lambda}} \mathbb{C}$.

The difference with the elliptic curve case is the lack of canonical Néron periods. Instead, there is a period map defined in [Kat04].

Definition A.3.4. Let $S(f)$ denote the Hecke eigenspace containing f in $S_2(Y_1(N))$. There is a canonical *period map* (see [Kat04, §4.10] for more details):

$$\mathrm{per}: \ S(f) \to V_{\mathbb{C}}(f)$$

For every $\gamma \in V_{\mathbb{C}}(f)$ there is an Euler system $\mathbf{z}_\gamma$ satisfying the following interpolation property:

Theorem A.3.5. ([Kat04, Theorem 12.5]) Let $\gamma \in V_{\mathbb{C}}(f)$. Then there is an Euler system $\mathbf{z}_\gamma$ such that, for every $F \in \Omega$ of conductor c and character ψ of $\mathrm{Gal}(F/\mathbb{Q})$, satisfies the relation

$$\sum_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \psi(\sigma)\mathrm{per}\big(\sigma(\exp^*(\mathrm{loc}_s^p(z_{\gamma,F})))\big)^{\psi(-1)} = L_{cp}(g, \chi, 1)\gamma^{\psi(-1)}$$

where, for some $\eta \in V_{K_g,\lambda}(f)$, $\eta^\pm$ denotes the projection of η to the eigenspace in which the complex conjugation acts by multiplication with $\pm$.

Remark A.3.6. Theorem A.3.2 can be recovered from Theorem A.3.5 after choosing a suitable γ using the Néron lattice of an elliptic curve.

A.4 Kolyvagin derivatives

The Kolyvagin derivative is an operator that constructs Kolyvagin systems for the canonical Selmer structure using an Euler system. More specifically, the Kolyvagin derivative can be seen as a map

$$D: \ \mathrm{ES}(T) \to \mathrm{KS}(T/p^k, \mathcal{F}^{\mathrm{can}})$$

from the Euler systems to the Kolyvagin systems of a p-adic Galois representation. We will sketch the construction of the Kolyvagin derivative under Assumption (BI), although it exists under less restrictive assumptions (see [Rub00, §4]).

We can call Kolyvagin derivative the following element in the group algebra.

Definition A.4.1. For $\ell \in \mathcal{P}(T/p^k)$, we can fix a generator $\sigma_\ell \in \mathcal{G}_\ell$. For all $j \leq k$, we define the j^{th} *Kolyvagin derivative* as the element

$$D_\ell^{(j)} := \sum_{i=1}^{p^j-1} i\sigma_\ell^i \in \mathbb{Z}_p[\mathcal{G}_\ell]$$

The most important property of the Kolyvagin derivative is the following identity. Denote by $\mathcal{H}_\ell^{(j)}$ the quotient of $\mathcal{G}_\ell$ of order p^j, and by $N_{\mathcal{H}_\ell^{(j)}}$ to the norm element in the group algebra of $\mathbb{Z}_p\big[\mathcal{H}_\ell^{(j)}\big]$. Note that $D_\ell^{(j)}$ can be seen as an element of $\mathbb{Z}_p\big[\mathcal{H}_\ell^{(j)}\big]$ too.The following identity can be obtained by a direct computation.

Proposition A.4.2. The Kolyvagin derivatives satisfy, as elements of $\mathbb{Z}_p\big[\mathcal{H}_\ell^{(j)}\big]$ the following holds:

$$(\sigma_\ell - 1)D_\ell^{(j)} \equiv p^j - N_{\mathcal{H}_\ell} \mod p^j$$

Notation A.4.3. When we write D_ℓ without any mention to the index j, we assume it is taken as the maximal j such that $\ell \in \mathcal{P}(T/p^j)$.

We can extend the definition of the Kolyvagin derivative to all vertices $n \in \mathcal{N}(T/p^k)$.

Definition A.4.4. Let $n = \ell_1 \dots \ell_s \in \mathcal{N}(T/p^k)$. For all $j \leq k$, the j^{th} *Kolyvagin derivative* operator is defined as

$$D_n^{(j)} = D_{\ell_1}^{(j)} \dots D_{\ell_s}^{(j)} \in \mathbb{Z}_p[\mathcal{G}_n]$$

Notation A.4.5. Similarly, when we write D_n without any mention to the index j, we are considering the product

$$D_n = D_{\ell_1} \dots D_{\ell_s} \in \mathbb{Z}_p[\mathcal{G}_n]$$

When applied to the classes in the Euler system, we obtain elements that are Galois invariant modulo powers of p.

Proposition A.4.6. ([Rub00, Lemma 4.4.2]) When $n \in \mathcal{N}(T/p^k)$, the element

$$D_n c_n \in H^1(K(n), T/p^k)^{\mathcal{G}_n}$$

where, in an abuse of notation, we are considering the image of the Kolyvagin derivative in the cohomology group $H^1(K(n), T/p^k)$, via the canonical projection map.

Using Proposition 2.1.6 and the inflation-restriction sequence, there restriction map induces an isomorphism

$$\text{res}: \ H^1(K, T/p^k) \to H^1(K(n), T/p^k)^{\mathcal{G}_n}$$

that can be used to construct the Kolyvagin derivative.

Definition A.4.7. Let $\mathbf{c}$ be an Euler system. For every square-free product $n \in \mathcal{N}(T/p^k)$, the Kolyvagin derivative is defined as

$$\widetilde{\kappa}_n := \text{res}^{-1}(D_n c_n)$$

The Kolyvagin derivatives satisfy the following:

Proposition A.4.8. ([Rub00, Theorems 4.5.1 and 4.5.4]) When $\mathbf{c}$ is an Euler system and $n \in \mathcal{N}(T/p^k)$, the Kolyvagin derivative satisfies that

$$\widetilde{\kappa}_{\mathfrak{n}} \in H^1_{(\mathcal{F}^{\text{can}})^n}(K, T/p^k)$$

Moreover, if $\ell \in \mathcal{P}(T/p^k)$ is a Kolyvagin prime not dividing n, then κ_n and $\kappa_{n\ell}$ satisfy the Kolyvagin system relation:

$$\phi_\ell^{\text{fs}}(\text{loc}_\ell(\widetilde{\kappa}_n)) = \text{loc}_\ell(\widetilde{\kappa}_{n\ell})$$

The Kolyvagin derivatives $\widetilde{\kappa}_n$ do not form a Kolyvagin system since they do not have the right local conditions at the primes dividing n. They form a *weak Kolyvagin system*, according to the terminology in [MR04]. However, they can be modified to obtain a Kolyvagin system as in Definition 3.1.2.

Assumption (Frob). Suppose that, for all $\ell \in \mathcal{P}$, $\text{Frob}_\ell^{p^k} - 1$ acts injectively on T.

We can modify the definition of $\widetilde{\kappa}_n$ to construct a Kolyvagin system. Let $\mathfrak{S}_n$ denote the set of permutations of the prime divisors of n. For every $\pi \in \mathfrak{S}_n$, we denote by $\operatorname{sign}(\pi)$ to the sign of the permutation and by d_π to the product of the prime divisors of n fixed by π.

For every pair of distinct prime divisors ℓ and q of n, consider the quantity $\omega_{\ell,q}$ defined as the unique element in $\mathbb{Z}/p^k$ such that

$$P_\ell(\mathrm{Frob}_q^{-1}) = (\sigma_\ell - 1)^{\omega_{\ell,q}} + I^2$$

where I is the augmentation ideal of $R[\mathcal{G}_\ell]$.

Definition A.4.9. ([MR04, (33)]) Under Assumption (Frob), for every $\mathfrak{n} \in \mathcal{N}$, the Kolyvagin class is defined as

$$\kappa_n := \sum_{\pi \in \mathfrak{S}_n} \left(\operatorname{sign}(\pi) \prod_{\ell \mid n/d_\pi} \omega_{\ell,\pi(\ell)} \widetilde{\kappa}_{d_\pi} \right)$$

The collection of classes κ_n forms a Kolyvagin system.

Theorem A.4.10. ([MR04, Theorem 3.2.4]) The classes $\{\kappa_n\}_{n \in \mathcal{N}}$ constructed in Definition A.4.9 form a Kolyvagin system.

Corollary A.4.11. For every $k \geq 1$, there exists a map

$$\mathrm{ES}(T) \to \mathrm{KS}(T/p^k, \mathcal{F}^{\mathrm{can}})$$

from the module of Euler systems to the Kolyvagin systems satisfying that κ_1 is the image of c_K via the canonical projection map

$$H^1(K,T) \to H^1(K, T/p^k)$$

By Theorem 5.2.19, we can consider the limit of these maps to construct ultra Kolyvagin systems.

Theorem A.4.12. There exists a canonical map

$$\Psi : \ \mathrm{ES}(T) \to \mathbf{KS}(T \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$$

satisfying that, if $\kappa = \Psi(\mathbf{c})$, then $\kappa_1 = c_K$.

The Euler and Kolyvagin system method can be also used to prove some cases of the Iwasawa main conjecture (see Definition A.2.3). In order to do that, we need to construct an ultra Kolyvagin system controlling the Selmer group $\mathbf{H}^1_{\mathcal{F}^*}(K_\infty, T^*)$, where K_∞/K is a $\mathbb{Z}_p$-extension. By Shapiro's lemma, that is equivalent to study the Galois representation

$$\mathbf{T} := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]] = T \otimes_{\mathbb{Z}_p} \Lambda$$

Similarly to the argument in §A.4, we construct Kolyvagin systems for the finite quotients of $\mathbf{T}$ described by

$$\mathbf{T}_k := \mathbf{T} \otimes_\Lambda \Lambda/(p^k, (X+1)^{p^k} - 1)$$

Note that, by Shapiro's lemma, the cohomology of $\mathbf{T}_k$ is

$$\mathbf{H}^1(K, \mathbf{T}_k) = \mathbf{H}^1(K, T/p^k \otimes \mathbb{Z}_p[\mathrm{Gal}(K_k/K)]] = \mathbf{H}^1(K_k, T/p^k)$$

where K_k/K is the subextension of K_∞/K of degree p^k.

Proposition A.4.13. ([Rub00, Lemma 4.4.2]) Let $n \in \mathcal{N}(\mathbf{T}_k)$. Then

$$D_n c_{K_k(n)} \in H^1(K_k, T/p^k)^{\mathcal{G}_n} = H^1(K, \mathbf{T}_k)^{\mathcal{G}_n}$$

We can then construct the derivative classes as in Definition A.4.7.

Definition A.4.14. Let $\mathbf{c}$ be an Euler system. For every vertex $n \in \mathcal{N}(T/p^k)$, the Kolyvagin class is defined as

$$\widetilde{\kappa}_n := \mathrm{res}^{-1}_{K_k(n)/K_k}(D_n c_{K_k(n)}) \in H^1(K, \mathbf{T}_k)$$

It can be shown that the collection $\{\widetilde{\kappa}_n\}_{n\in\mathcal{N}}$ forms a weak Kolyvagin system. In order to obtain a Kolyvagin system, we need to use the modification in Definition A.4.9.

Theorem A.4.15. ([MR04, Theorem 5.3.3]) Define

$$\kappa_n := \sum_{\pi \in \mathfrak{S}_n} \left(\mathrm{sign}(\pi) \prod_{\ell | n/d_\pi} \omega_{\ell,\pi(\ell)} \widetilde{\kappa}_{d_\pi} \right)$$

Then $\{\kappa_n\}_{n\in\mathbb{N}}$ is a Kolyvagin system in $\mathrm{KS}(\mathbf{T}_k, \mathcal{F}^{\mathrm{can}})$.

Corollary A.4.16. For every $k \geq 1$, there exists a map

$$\mathrm{ES}(T) \to \mathrm{KS}(T/p^k \otimes \mathbb{Z}_p[[\mathrm{Gal}(K_k/K)]], \mathcal{F}^{\mathrm{can}})$$

from the module of Euler systems to the Kolyvagin systems satisfying that κ_1 is the image of c_{K_k} via the canonical projection map

$$H^1(K_k, T) \to H^1(K_k, T/p^k) \cong H^1(K, T/p^k \otimes \mathbb{Z}_p[[\mathrm{Gal}(K_k/K)]])$$

By Theorem 5.2.19, we can consider the limit of these maps to construct ultra Kolyvagin systems.

Theorem A.4.17. There exists a canonical map

$$\Psi_\infty : \ \mathrm{ES}(T) \to \mathbf{KS}(T \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$$

satisfying that, if $\kappa = \Psi_\infty(\mathbf{c})$, then $\kappa_1 = c_{K_\infty}$.

Remark A.4.18. By Theorem 6.3.19, an Euler system $\mathbf{c}$ satisfies the Iwasawa main conjecture if and only if $\Psi_\infty(c)$ is a primitive Kolyvagin system in $\mathbf{KS}(T \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$.

Remark A.4.19. If Ψ and Ψ_∞ are the maps constructed in Theorems A.4.12 and A.4.17 and Π is the projection map from Proposition 6.5.1, then $\Psi = \Pi \circ \Psi_\infty$.

Remark A.4.20. The Kolyvagin derivative process can be generalised to the case in which $\chi(\mathcal{F}^{\mathrm{can}}) = r > 1$, obtaining a map

$$\mathrm{ES}_r(T) \to \mathbf{KS}(\mathcal{F}^{\mathrm{can}})$$

where $\mathrm{ES}_r(T)$ denotes the module of Euler systems of rank r. See [Büy10] or [BS21] for more details.

We conclude this section with a proof of Theorem 6.5.5.

Proof of Theorem 6.5.5. As shown in [MR04, Lemma 5.3.1],

$$H^1_{\mathrm{IW}}(K, T) = H^1_{\mathcal{F}^{\mathrm{can}}}(K, T)$$

Let Ψ_∞ be the map constructed in Theorem A.4.17. Since $\Psi_\infty(\mathbf{c})_1 = c_{K_\infty}$, the Iwasawa main conjecture for $\mathbf{c}$ tells that

$$\mathrm{ind}\big(\Psi_\infty(\mathfrak{c})_1\big) \approx \mathrm{Fitt}^0_\Lambda\big(\mathbf{H}^1_{\mathcal{F}^*}(K, T^*)^\vee\big)$$

From Theorem 6.3.18, it follows that $\Psi_\infty(\mathbf{c})$ is a primitive Kolyvagin system in $\mathbf{KS}(T \otimes \Lambda, \mathcal{F}^{\mathrm{can}})$. From Corollary 6.5.3 and Remark A.4.19, we can conclude that $\Psi(\mathbf{c})$ is also primitive, where Ψ is the map constructed in Theorem A.4.12. $\square$

Appendix B

Algebraic Preliminaries

In this appendix, we present some of the algebraic preliminaries required for this thesis. It is important to highlight the rank reduction maps in §B.4.1, which are central objects in the construction of Kolyvagin, Stark and cartesian systems.

B.1 Self-injective rings

This section describes the basic definitions and properties of self-injective rings.

Definition B.1.1. A self-injective (commutative) ring R is a commutative ring which is injective as an R-module.

Remark B.1.2. If R is a self-injective ring and I is an ideal of R, then every map in $\mathrm{Hom}(I, R)$ extends to a map in $\mathrm{Hom}(R, R)$.

Proposition B.1.3. If R is a self-injective ring, the functor from the category of R-modules to itself given by

$$M \mapsto M^+ := \mathrm{Hom}(M, R)$$

is exact.

Proof. Such functor is left exact for all rings R (see [AM69, Proposition 2.9]). The exactness follow from the definition of injective module. $\square$

Proposition B.1.4. Let M be a finitely generated R-module. The canonical map

$$\Phi_M : \ M \to M^+{+} = \mathrm{Hom}(M, R) : \ a \mapsto (\varphi \mapsto \varphi(a))$$

is an isomorphism.

Proof. Choose a free F module and another R-module N admitting a short exact sequence

$$0 \longrightarrow N \xrightarrow{\mu} F \xrightarrow{\varepsilon} M \longrightarrow 0$$

Since Φ_F is clearly surjective and $\varepsilon^{++}: F^{++} \twoheadrightarrow M^{++}$ is surjective, the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\mu"] \arrow[d, "\Phi_N"] & F \arrow[r, "\varepsilon"] \arrow[d, tail, two heads, "\Phi_F"] & M \arrow[r] \arrow[d, "\Phi_M"] & 0 \\ 0 \arrow[r] & N^{++} \arrow[r, "\mu^{++}"] & F^{++} \arrow[r, "\varepsilon^{++}"] & M^{++} \arrow[r] & 0 \end{tikzcd}$$

Since Φ_F is surjective, the snake lemma implies that Φ_M is surjective. The same argument, with a different exact sequence, implies that Φ_N is also surjective. Then the snake lemma and the injectivity of Φ_F imply the injectivity of Φ_M. □

The technical advantage of self-injective rings is that annihilators define an order-reversing involution in the set of ideals of R.

Definition B.1.5. Let R be a commutative ring and let I be an ideal of R. The *annihilator* of I is the ideal defined as

$$R[I] = \{x \in R : \ xI = 0\}$$

Remark B.1.6. Whenever $R[I]$ is a principal ideal, it is isomorphic to R/I. In general, $R[I]$ is isomorphic to $\mathrm{Hom}(R/I, R)$.

Proposition B.1.7. Let R be a self-injective ring and let I be a finitely generated ideal of R. Then

$$R\big[R[I]\big] = I$$

Proof. By Remark B.1.6, there is a short exact sequence

$$0 \longrightarrow R[I] \longrightarrow R^+ \longrightarrow I^+ \longrightarrow 0$$

Applying Remark B.1.6 to R/I, we have that

$$R\big[R[I]\big] = \mathrm{Hom}(R/R[I], R) = \mathrm{Hom}(I^+, R) = I^{++} = I$$

where the last equality follows from B.1.4.

By definition, it is straightforward to check that $I \subset R\big[R[I]\big]$. Conversely, let $x \in R\big[R[I]\big]$. □

Remark B.1.8. The association $I \mapsto R[I]$ is an order-reversing involution in the set of ideals of m.

Proposition B.1.9. If R is a noetherian self-injective ring, then R is artinian.

Proof. Consider a descending chain of ideals $(I_i)_{i\in\mathbb{N}}$. Then, their annihilators $\big(\mathrm{Ann}(I_i)\big)_{i\in\mathbb{N}}$ form an ascending sequence, which stabilizes because R is noetherian. By Proposition B.1.7, $(I_i)_{i\in\mathbb{N}}$ also stabilizes. $\square$

In general, self-injective rings can fail to be noetherian. It is possible to find a counterexample by considering direct products of self-injective rings.

Proposition B.1.10. Let $\{R_i : i \in I\}$ be a collection of self-injective rings. Then the direct product

$$R := \prod_{i\in I} R_i$$

is also a self-injective ring.

Proof. Consider an injection $M \hookrightarrow N$ of R-modules. For M and N, there is an identification fitting in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(N,R) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_R(M,R) \\
\prod_{i\in I}\mathrm{Hom}_{R_i}(N,R_i) \arrow[r, two heads] & \prod_{i\in I}\mathrm{Hom}_{R_i}(M,R_i)
\end{tikzcd}$$

where the bottom map is surjective because R_i is injective for all $i \in I$. Hence, the top row is also surjectie, which shows that R is self-injective. $\square$

Remark B.1.11. If $(R,\mathfrak{m})$ is local, there is a 'minimal ideal' $R[\mathfrak{m}]$ which is contained in every non-zero ideal of R. In particular, it is a principal ideal. Also note that all its elements are divisible by any non-zero element of R.

We want to give a criterion to determine whether a local, artinian ring $(R/\mathfrak{m})$ is self-injective. If k-denotes the residue field of R, self-injectivity implies by Remark B.1.11 that

$$\mathrm{Hom}_k(k,R) = \mathrm{Hom}(R/\mathfrak{m},R) \cong R[\mathfrak{m}]$$

is a one dimensional k-vector space. We will use the converse to prove the self-injectivity of rings.

Proposition B.1.12. Let $(R, \mathfrak{m})$ be an artinian, local ring with residue field k. If $\mathrm{Hom}_k(k, R)$ is a one-dimensional k-vector space.

Proof. The proof considers the injective hull $E(R/\mathfrak{m}^k)$ as an R-module. Since it $E(R/m^k)$, we can extend the map definitional map

$$R/\mathfrak{m} \hookrightarrow E(R/m^k)$$

to a map $\psi : R \to E(\mathfrak{m}^k)$ such that the following diagram is commutative

$$\begin{tikzcd}
R/\mathfrak{m} \arrow[r] \arrow[dr] & R \arrow[d] \\
 & E(R/\mathfrak{m})
\end{tikzcd}$$

Since $R[\mathfrak{m}]$ is one-dimensional, the horizontal map induces an isomorphism $R/\mathfrak{m} \cong R[\mathfrak{m}]$. Since the diagonal map is injective, we can deduce that

$$\ker(\psi) \cap R[\mathfrak{m}] = 0$$

which implies that ψ is injective. By Matlis duality, $\ell_R(E(R/m)) = \ell_R(R)$, to ψ is an isomorphism and, hence, R is self-injective. $\square$

We can characterise the self-injective quotients of a self-injective ring.

Proposition B.1.13. Let R be a self-injective ring and let I be an ideal of R. Then R/I is a self-injective ring if and only if $R[I]$ is a principal ideal.

Proof. Recall that

$$R[I] = \ker\Big(R^+ \to I^+\Big) = (R/I)^+$$

By Nakayama's lemma, $R[I]$ is principal if and only if $R[I]/\mathfrak{m}$ is a one-dimensional vector space. By the identification above, that is equivalent to $(R/I)[\mathfrak{m}]$ being one dimensional, which is equivalent, by Proposition B.1.12, to the self-injectivity of R. $\square$

Proposition B.1.12 can be used to prove that the following examples are self-injective rings.

- For every $k \in \mathbb{N}$, Z/p^k is a self-injective ring. More generally, Chinese remainder theorem implies that Z/n is a self-injective ring for every $n \in \mathbb{Z}$.
- If G is a finite, abelian group, the group algebra $R = \mathbb{Z}/nZ[G]$ is a self-injective ring. Using Chinese remainder theorem, representation theory to split into character parts and Proposition B.1.10, we are reduced to the case where n is a power of p, say p^k and G is an abelian cyclic p-group, say of order p^n. Then R is local and $R[m]$ is generated by $p^{k-1}N$, where N is the norm element. Then R is self-injective by Proposition B.1.12.
- If $R = \Lambda/(p^k, f)$, where Λ is the Iwasawa algebra and f is a distinguished polynomial, then R is a self-injective ring by Proposition B.1.12, since $R[m]$ is one-dimensional and generated by $p^{k-1}X^{\deg(f)-1}$.

B.2 Fitting ideals

In this section, we give an introduction to the theory of Fitting ideals. They can be seen as ideals encoding the structure of a (finitely presented) module. When the ring R is simple, the sequence of Fitting ideals can determine the module up to isomorphism, but that is not true in general.

Definition B.2.1. Let M be a finitely presented R-module. Choose a resolution

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

where the map $R^n \to R^m$ is represented by the matrix A. For every $i \geq 0$, we define the i^{th} Fitting ideal $\mathrm{Fitt}^i_R(M)$ as the ideal in R generated by the minors of size $(m-i)$ of A.

Remark B.2.2. The i^{th} Fitting ideal coincides with the image of the following map (see Definition B.4.1 below), induced by the matrix A:

$$\mathrm{Fitt}^i_R(M) = \mathrm{Im}\left(\bigwedge^{m-i} R^n \to \bigwedge^{m-i} R^m\right)$$

It can be shown that Fitting ideals are well defined.

Proposition B.2.3. ([Eis95, Corollary 20.4]) The Fitting ideals $\mathrm{Fitt}^i_R(M)$ are independent of the chosen resolution.

The following results can be useful.

Proposition B.2.4. Let M be an R-module and let $R \to S$ be a ring-homomorphism. Then

$$\mathrm{Fitt}^i_S\big(M \otimes_R S\big) = \mathrm{Fitt}^i_R(M) \otimes S \ \forall i \in \mathbb{Z}_{\geq 0}$$

Proof. Consider a resolution of M given by

$$R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

Since the tensor product is a right exact functor, we can obtain a resolution of $M \otimes S$:

$$S^n \xrightarrow{A \otimes S} S^m \longrightarrow M \otimes_R S \longrightarrow 0$$

Then the equality follows from the definition of Fitting ideals. $\square$

Proposition B.2.5. Assume that R is a domain and M is a torsion R-module of projective dimension at most one. Then $\mathrm{Fitt}^0_R(M)$ is a principal ideal.

Proof. Since the projective dimension of M is at most one, it admits a resolution

$$0 \longrightarrow R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0$$

Since the rank is an additive function, then $m = n$, so $\mathrm{Fitt}^0(M)$ is generated by $\det(A)$. $\square$

When R is a discrete valuation ring, the Fitting ideals determine the module up to isomorphism.

Proposition B.2.6. Let R be a discrete valuation ring and let M be a finitely generated R-module. Then the Fitting ideals determine M up to isomorphism and can be described explicitly.

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. The structure theorem of finitely generated modules over principal ideal domains implies that there are integers r, s and $\alpha_1 \geq \ldots \geq \alpha_s$ such that

$$M \cong R^r \times R/\mathfrak{m}^{\alpha_1} \times \cdots \times R/\mathfrak{m}^{\alpha_s}$$

Then M admits a resolution

$$R^{r+s} \xrightarrow{A} R^{r+s} \longrightarrow M \longrightarrow 0$$

where the matrix A is given by

$$A = \left(\begin{array}{c:ccc} \mathbf{0}_{r\times r} & & \mathbf{0}_{s\times r} & \\ \hdashline & \pi^{\alpha_1} & & \\ \mathbf{0}_{r\times s} & & \ddots & \\ & & & \pi^{\alpha_s} \end{array}\right)$$

where π is a generator of $\mathfrak{m}$. We can then compute,

- $i \in \{0, \dots, r-1\} \Rightarrow \mathrm{Fitt}^i_R(M) = (0)$
- $j \in \{0, \dots, s-1\} \Rightarrow \mathrm{Fitt}^{r+j}_R(M) = \prod_{k=j+1}^{s} \mathfrak{m}^{i_k} = \mathfrak{m}^{\sum_{k=j+1}^{s} i_k}$
- $i \geq r+s \Rightarrow \mathrm{Fitt}^i(M) = (1)$.

The Fitting ideals determine M up to isomorphism, since

- r is the minimum i such that $\mathrm{Fitt}^i(M) \neq 0$.
- For $i \geq 0$, $\alpha_i = \mathrm{Fitt}_{r+i+1}(M)\mathrm{Fitt}_{r+i}(M)^{-1}$.

□

As a consequence, we can prove an additive property of the Fitting ideals over discrete valuation rings.

Corollary B.2.7. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of modules over a quotient of a discrete valuation ring R. Then

$$\mathrm{Fitt}^0_R(B) = \mathrm{Fitt}^0_R(A)\mathrm{Fitt}^0_R(C)$$

Remark B.2.8. By Proposition B.2.4, when R is a quotient of a discrete valuation ring, the Fitting ideals of an R-module have the structure defined in the proof of Proposition B.2.6.

For modules over more complicated rings, the Fitting ideals give partial information about the module. Particularly, they describe Iwasawa modules up to pseudo-isomorphism.

Proposition B.2.9. Let $\Lambda = \mathbb{Z}_p[[X]]$ be the Iwasawa algebra, and let M be a finitely generated Λ-module. Then the Fitting ideals determine Λ up to pseudo-isomorphism.

Proof. By the structure theorem of finitely generated modules, there is a pseudo-isomorphism

$$M \approx \Lambda^r \times \prod_\beta \prod_{i=1}^{t_\beta} \Lambda/(\beta)^{\alpha_{\beta,i}}$$

where β runs through the height one prime ideals of Λ, and t_β and the α_i are non-negative integers such that $t_\beta = 0$ for all but finitely many β.

By Proposition B.2.4 applied to the homomorphism $\Lambda \to \Lambda_\beta$, the localization at any prime ideal β induces an isomorphisms

$$\mathrm{Fitt}^i_\Lambda(M)_\beta = \mathrm{Fitt}^i_{\Lambda_\beta}(M_\beta) = \mathrm{Fitt}^i\Big(\Lambda^r_\beta \times \prod_{i=1}^{t_\beta} \Lambda_\beta/(\beta)^{\alpha_i}\Big)$$

As in the proof of Proposition B.2.6, we can compute

- $i \in \{0, \ldots, r-1\} \Rightarrow \mathrm{Fitt}^i_{\Lambda_\beta}(M_\beta) = (0)$
- $j \in \{0, \ldots, s-1\} \Rightarrow \mathrm{Fitt}^{r+j}_{\Lambda_\beta}(M_\beta) = \prod_{k=j+1}^{t_\beta} \beta^{i_k} = \beta^{\sum_{k=j+1}^{s} i_k}$
- $i \geq r+s \Rightarrow \mathrm{Fitt}^i(M) = (1)$.

We can also see similarly that the Fitting ideals recover M_β up to isomorphism of Λ_β-modules. Since that holds for all height one prime ideals, the Fitting ideals can recover Λ up to pseudo-isomorphism. □

Localising at every height one prime ideal β, we can use Corollary B.2.7 to prove an analogue for Iwasawa modules up to pseudo-equality (see Notation 6.2.9).

Corollary B.2.10. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of modules over a quotient of a discrete valuation ring R. Then

$$\mathrm{Fitt}^0_\Lambda(B) \approx \mathrm{Fitt}^0_\Lambda(A)\mathrm{Fitt}^0_\Lambda(C)$$

Proof. For every height one prime ideal β, there is an exact sequence

$$0 \longrightarrow A_\beta \longrightarrow B_\beta \longrightarrow C_\beta \longrightarrow 0$$

By Corollary B.2.7,

$$\mathrm{Fitt}^0_{\Lambda_\beta}(B_\beta) = \mathrm{Fitt}^0_\Lambda(A_\beta)\mathrm{Fitt}^0_\Lambda(C_\beta)$$

Since the localization and the Fitting ideals commute, by Proposition B.2.4, we get that

$$\mathrm{Fitt}^0_\Lambda(B)_\beta = \mathrm{Fitt}^0_\Lambda(A)_\beta\mathrm{Fitt}^0_\Lambda(C)_\beta$$

Since that is true for any height one prime ideal β, we can conclude that

$$\mathrm{Fitt}^0_\Lambda(B) \approx \mathrm{Fitt}^0_\Lambda(A)\mathrm{Fitt}^0_\Lambda(C)$$ □

Fitting ideals can determine more than the structure of the module up to pseudo-isomorphism. In fact, they can determine whether a torsion Iwasawa module contains finite submodules.

Proposition B.2.11. Let M be a torsion Iwasawa module. Then $\mathrm{Fitt}^0_\Lambda(M)$ is a principal ideal if and only if M contains no finite modules.

Proof. Assume M contains no finite modules. Then [NSW00, Proposition 5.5.3.(iv)] implies that its projective dimension is at most one, so Proposition B.2.5 implies that $\mathrm{Fitt}^0_\Lambda(M)$ is principal.

Conversely, assume that M contains a finite submodule. Denote by $T_0(M)$ the maximal finite submodule of M.

We can construct a projective resolution of M building from projective resolutions of M and $M/T_0(M)$, obtaining a diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \Lambda^a \arrow[r, "A"] \arrow[d] & \Lambda^x \arrow[r] \arrow[d] & T_0(M) \arrow[r] \arrow[d] & 0 \\
 & \Lambda^b \arrow[r, "B"] \arrow[d] & \Lambda^y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda^c \arrow[r, "C"] \arrow[d] & \Lambda^z \arrow[r] \arrow[d] & M/T_0(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $M/T_0(M)$ contains no finite submodules, [NSW00, Proposition 5.5.3.(iv)] implies that its projective dimension is at most one, which leads to the left exactness of the bottom row. Since it is a torsion module, we can assume that $c = z$, so C is a square matrix.

By the construction of the projective resolutions, the block representation of the matrix B is

$$N = \left(\begin{array}{c:c} A & * \\ \hdashline 0_{x\times c} & C \end{array}\right)$$

Since $y = x + z$, every non-zero minor of size i has to be a product of $\det(C)$ with a minor of size x in A. Hence,

$$\mathrm{Fitt}^0_\Lambda(M) = \mathrm{Fitt}^0_\Lambda\big(T_0(M)\big)\mathrm{Fitt}_0\big(M/T_0(M)\big)$$

Note that $\mathrm{Fitt}_0\big(M/T_0(M)\big)$ is generated by $\det(C)$ and, since the quotient $T/T_0(M)$ has rank zero, it is a non-zero principal ideal.

Hence the proof will be completed once we show that $\mathrm{Fitt}^0_\Lambda\big(T_0(M)\big)$ is a (non-principal) ideal strictly contained in Λ with finite index.

But every minor of size x cannot be a unit in Λ since, otherwise, the map generated by A would be surjective and $T_0(M) = 0$, contradicting the assumption. Therefore,

$$\mathrm{Fitt}^0_\Lambda\big(T_0(M)\big) \subset \mathfrak{m}$$

Then we can conclude with Proposition B.2.9 since for every height one prime ideal β

$$\mathrm{Fitt}^0_\Lambda\big(T_0(M)\big)_\beta = \mathrm{Fitt}^0_{\Lambda_\beta}\big(T_0(M)_\beta\big) = \Lambda_\beta$$

$\square$

B.3 Extension of Iwasawa modules

This section includes two basic results about the extension groups of Iwasawa modules. They will be required to construct the rank reduction maps in Proposition B.4.8 below.

Lemma B.3.1. Let M be a submodule of Λ^s. Then $\mathrm{Ext}^1(M, \Lambda)$ is finite.

Proof. For every Λ-module M, we have defined $M^+ := \mathrm{Hom}(M, \Lambda)$. There is a canonical map

$$\Phi : \; M \to M^{++}, \; m \mapsto (\varphi \in M^+ \mapsto \varphi(a))$$

Let $T_1(M) = \ker \Phi$. Since M is contained in Λ^s, we claim that $T_1(M) = 0$. Indeed, fix an inclusion $\iota : M \hookrightarrow \Lambda^s$ and denote by $\pi^i : \Lambda^s \to \Lambda$ the projection at the i^{th} coordinate. Let $m \in M \setminus \{0\}$. Then there is some $j \in \{1, \ldots, s\}$ such that the j^{th} coordinate of $\iota(m)$ is non-zero. Then

$$\Phi(m)(\pi^j \circ \iota) = \pi_j(\iota(m)) \neq 0$$

Thus $m \notin T_1(M)$ and the proof of the claim is complete.

By [NSW00, Corollary 5.5.9], $\mathrm{Ext}^1(M, \Lambda)$ is finite. $\square$

Lemma B.3.2. Let $\varphi : N \to M$ be a pseudo-isomorphism of Λ-modules. Then the dual map $\varphi^+ : M^+ \to N^+$ is an isomorphism.

Proof. Let C and D be finite Λ modules fitting in the exact sequence

$$0 \longrightarrow C \longrightarrow M \longrightarrow N \longrightarrow D \longrightarrow 0$$

Let K be the kernel of the map $N \to D$. Dualizing, there is an exact sequence

$$D^+ \longrightarrow N^+ \longrightarrow K^+ \longrightarrow \mathrm{Ext}^1(K, \Lambda)$$

Since D is finite, then both D^+ and $\mathrm{Ext}^1(D, \Lambda)$ vanish. Indeed, $D^+ = 0$ since Λ does not contain elements of finite order and $\mathrm{Ext}^1(D, \Lambda) = 0$ by [NSW00, Corollary 5.5.4]. Hence the dual map $N^+ \to D^+$ is an isomorphism. There is another exact sequence

$$0 \longrightarrow C \longrightarrow M \longrightarrow K \longrightarrow 0$$

Since C is finite, $C^+ = 0$, so the dual map is $K^+ \to M^+$ is an isomorphism. Thus φ^+ is an isomorphism. $\square$

B.4 Exterior powers

The goal of this section is to introduce the necessary theory of exterior biduals. Its most important part is the rank reduction maps, which are central objects in this thesis.

Exterior biduals can be defined for modules over any ring R, but the rank reduction maps are only defined for some well-behaved rings.

Definition B.4.1. Let R be a ring, let M be an R-module and let $r \in M$. The r^{th} *exterior power*

$$\bigwedge^r M$$

is defined as the quotient of the tensor product $M^{\otimes r}$ by the submodule generated by the elements of the form

$$m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_j \otimes \cdots \otimes m_r + m_1 \otimes \cdots \otimes m_j \otimes \cdots \otimes m_i \otimes \cdots \otimes m_r$$

Definition B.4.2. If F is a free R-module of rank s, the determinant of F is defined as

$$\det(F) = \bigwedge^s F$$

We now state two basic properties of the determinant.

Proposition B.4.3. Let $A \subset B$ be free R-modules such that B/A is also free. Then there is a canonical isomorphism

$$\det(A) \otimes \det(B/A) \cong \det(B)$$

Proposition B.4.4. Let A be a free R-module. There is a canonical isomorphism

$$\det(A) \otimes \det(A^+) \cong R$$

Definition B.4.5. Let R be a ring, let M be an R-module and let $r \in M$. The r^{th} *exterior bidual* is defined as

$$\bigcap^r M = \left(\bigwedge^r M^+\right)^+$$

The properties of exterior biduals will be deduced assuming certain properties of the ring R.

B.4.1 Rank reduction of exterior powers

The goal of this section is to define the rank reduction maps between exterior biduals. This map was constructed in [Sak18, Lemma 4.8] and [BS21, Proposition A.2] for modules over a self-injective rings R (see §B.1), but we generalise it for modules over discrete valuation rings and Iwasawa modules.

Proposition B.4.6. Consider the following exact sequence of R-modules over a self-injective ring R:

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F$$

where F is a free R-module of rank s. If $r \geq s$ is a natural number, there is a canonical map

$$\phi_{N,M} : \left(\bigcap^{r} N \right) \otimes \det(F^{+}) \to \bigcap^{r-s} M$$

Proof. By Proposition B.1.3, there is another exact sequence

$$F^{+} \xrightarrow{\varepsilon^{+}} N^{+} \xrightarrow{\mu^{+}} M^{+} \longrightarrow 0$$

For every $m \in M^{+}$, choose a lift $n \in N^{+}$. With this choices, there is a set-theoretic section

$$\widetilde{\mu^{+}} : \bigwedge^{r-s} M^{+} \to \bigwedge^{r-s} N^{+}$$

This section can be used to construct a well defined, independent of the choice of $\widetilde{\mu^{+}}$, R-homomorphism

$$\phi^{+}_{N,M} : \left(\bigwedge^{r-s} M^{+} \right) \otimes \det(F^{+}) \to \bigwedge^{r} N^{+}, \ m \otimes f \mapsto \widetilde{\mu^{+}}(m) \otimes \varepsilon^{+}(f)$$

The dual map can be expressed as

$$\phi_{N,M} : \left(\bigcap^{r} N \right) \otimes \det(F^{+}) \to \bigcap^{r-s} M$$ ◻

This map can be also constructed when R is a discrete valuation ring. The main technical problem in this case is that the dualizing functor is no longer exact, but that issue can be overcome using the fact that all submodules of free modules are themselves free.

Proposition B.4.7. Consider the following exact sequence of R-modules over a discrete valuation ring R:

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F$$

where F is a free R-module of rank s. If $r \geq s$ is a natural number, there is a canonical map

$$\phi_{N,M} : \left(\bigcap^{r} N \right) \otimes \det(F^+) \to \bigcap^{r-s} M$$

Proof. The image of ε is a submodule of F so, in particular, it is torsion-free. Therefore, it is a free R-module, so $\mathrm{Ext}^1(\mathrm{Im}(\varepsilon), R) = 0$. Hence there is an exact sequence

$$0 \longrightarrow \mathrm{Im}(\varepsilon)^+ \longrightarrow N^+ \longrightarrow M^+ \longrightarrow 0$$

We can generalise the argument in Proposition B.4.6 to construct the map $\phi_{N,M}$. For every $m \in M^+x$, choose a lift $n \in N^+$, which also defines a set-theoretic section

$$\alpha : \bigwedge^{r-s} M^+ \to \bigwedge^{r-s} N^+$$

We can then define the map

$$\bigwedge^{r-s} M^+ \otimes \det(F^+) \to \bigwedge^{r} N^+ : \; m \otimes x \mapsto \alpha(m) \wedge \varepsilon^+(x) \qquad \text{(B.1)}$$

We must show that this map is independent of the section chosen. That is clear when $r = s$, so we can assume that $r > s$. If

$$\beta : \bigwedge^{r-s} M^+ \to \bigwedge^{r-s} N^+$$

is a different section, then

$$\big(\alpha(m) - \beta(m)\big) \wedge x \in \bigwedge^{r} \mathrm{Im}(\varepsilon)^+$$

However, we know that $\mathrm{Im}(\varepsilon)^+$ is a free R-module of rank at most s, so the exterior power vanishes. That proves that the map in (B.1) is well defined. Then $\phi_{N,M}$ is constructed as the dual map. □

Finally, we construct this map for biduals of modules over the Iwasawa algebra. This case is more complicated than the previous ones since the submodules of free, finitely generated Λ-modules are no longer free and might have non-trivial extensions. However, we can use Lemmas B.3.1 and B.3.2 to construct the map between biduals.

Proposition B.4.8. Consider the exact sequence of Λ-modules

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F$$

where F is a torsion free Iwasawa module of rank s. If $r \geq s$ is a natural number, there is a canonical map

$$\phi_{N,M}: \left(\bigcap^{r} N\right) \otimes \det(F^+) \to \bigcap^{r-s} M$$

Proof. The structure theorem of finitely generated Iwasawa modules implies that F is contained with finite index in a free module of rank s. Then Lemma B.3.2 implies that F^+ is a free Λ-module of rank s.

Let I be the image of ε. Since I is contained in a free Iwasawa module, Lemma B.3.1 implies that $\mathrm{Ext}^1(I,\Lambda)$ is finite. There is an exact sequence

$$0 \longrightarrow I^+ \xrightarrow{\varepsilon^+} N^+ \xrightarrow{\mu^+} M^+ \longrightarrow \mathrm{Ext}^1(I,\Lambda)$$

Call J the image of μ^+, which has finite index in M^+ because of the finiteness of $\mathrm{Ext}^1(I,\Lambda)$. The exact sequence

$$0 \longrightarrow I^+ \longrightarrow N^+ \longrightarrow J \longrightarrow 0$$

induces, as in Proposition B.4.7, a canonical map

$$\bigwedge^{r-s} J \otimes \det(F^+) \to \bigwedge^{r} N^+$$

The fact that this map is well-defined independently of the choice of the section deserves some comment. Similarly, we can assume that $r > s$ and let

$$\alpha, \beta: \bigwedge^{r-s} J \to \bigwedge^{r-s} N^+$$

be two sections. For any $j \in \bigwedge^{r-s} J$, we have that

$$\big(\alpha(j) - \beta(j)\big) \wedge x \in \varepsilon\left(\bigwedge^{r} I\right)$$

However, we know that I^+ is a torsion-free, reflexive module, so it has to be free. Since its rank is at most s, the exterior power vanishes, so the map is well defined. The dual of this map induces the following:

$$\left(\bigcap^{r+s} N^+\right) \otimes \det(F^+) \to \mathrm{Hom}\left(\bigwedge^r J, \Lambda\right)$$

Since $\bigwedge^r J$ has finite index in $\bigwedge^r N^+$, Lemma B.3.2 implies that their duals are equal, so the above map can be rewritten as

$$\left(\bigcap^{r+s} N\right) \otimes \det(F^+) \to \bigcap^r M \qquad \square$$

When F is canonically isomorphic to R, then $\det(F^+)$ is also canonically identified with R and the rank reduction map can be written as follows.

Corollary B.4.9. Let R be either a self-injective ring, a discrete valuation ring, or the Iwasawa algebra. Assume we have the following exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} R^s$$

for some natural number s. If $r \geq s$ is another natural number, there is a canonical map

$$\phi_{N,M} : \bigcap^r N \to \bigcap^{r-s} M$$

The rank reduction map behaves naturally in the following sense.

Proposition B.4.10. (Functoriality of the rank reduction) Let R be either a self-injective ring, a discrete valuation ring or the Iwasawa algebra and consider the following commutative diagram of R-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "\mu_1"] \arrow[d, "\alpha"] & N_1 \arrow[r, "\varepsilon_1"] \arrow[d, "\beta"] & F_1 \arrow[d, "\gamma"] \\ 0 \arrow[r] & M_2 \arrow[r, "\mu_2"] & N_2^{\varepsilon_2} \arrow[r] & F_2 \end{tikzcd}$$

where F_1 and F_2 are free modules of rank s. Then, for every $r \geq 0$, the following diagram is commutative:

$$\begin{tikzcd} \bigcap^{r+s} N_1 \arrow[r, "\phi_{N_1,M_1}"] \arrow[d, "\bigcap^{r+s}\beta"] & \bigcap^r M_1 \arrow[d, "\bigcap^r \alpha"] \\ \bigcap^{r+s} N_2 \arrow[r, "\phi_{N_2,M_2}"] & \bigcap^r M_2 \end{tikzcd}$$

$$\begin{tikzcd}
\bigcap^{r+s} N_1 \arrow[r, "\phi_{N_1,M_1}"] \arrow[d, "\bigcap^{r+s}\beta"] & \bigcap^{r} M_1\alpha \arrow[d, "\bigcap^{r}\alpha"] \\
\bigcap^{r+s} N_2 \arrow[r, "\phi_{N_2,M_2}"] & \bigcap^{r} M_2
\end{tikzcd}$$

where the horizontal maps are constructed in Propositions B.4.6, B.4.7 and B.4.8.

Proof. Dualising, we obtain a diagram

$$\begin{tikzcd}
F_2^+ \arrow[r, "\varepsilon_2^+"] \arrow[d, "\gamma^+"] & N_2^+ \arrow[r, "\mu_2^+"] \arrow[d, "\beta^+"] & M_2^+ \arrow[d, "\alpha^+"] \\
F_1^+ \arrow[r, "\varepsilon_1^+"] & N_1^+ \arrow[r, "\mu_1^+"] & M_1^+
\end{tikzcd}$$

in which the rows are no longer exact.

Define $J_i := \mathrm{Im}(\mu_i^+ : N_i^+ \to M_i^+)$ for $i = 1, 2$. Note that the commutativity gives the existence of a well-defined map

$$\alpha^+ : J_2 \to J_1$$

It is possible to choose sections $\widetilde{\mu_i^+} : J_i \to N_i^+$ such that the following diagram commutes:

$$\begin{tikzcd}
J_2 \arrow[r, "\widetilde{\mu_2^+}"] \arrow[d, "\alpha^+"] & N_2^+ \arrow[d, "\beta^+"] \\
J_1 \arrow[r, "\widetilde{\mu_1^+}"] & N_1^+
\end{tikzcd}$$

Then, it follows from the construction process than the following diagram commutes

$$\begin{tikzcd}
\bigwedge^{r-s} J_2 \otimes \det(F_2^+) \arrow[r] \arrow[d, "\wedge^{r-s}\alpha^+ \otimes \wedge^s \gamma^+"] & \bigwedge^{r} N_2^+ \arrow[d, "\wedge^r \beta^+"] \\
\bigwedge^{r-s} J_1 \otimes \det(F_1^+) \arrow[r] & \bigwedge^{r} N_1^+
\end{tikzcd}$$

The proposition then follows by considering the dual of this diagram. $\square$

B.4.2 Commutative property of the rank reduction

The behaviour of the composition of two of these maps is studied in the next proposition.

Proposition B.4.11. Consider the following exact sequences of R modules, where R is either a self-injective ring, a discrete valuation ring or the Iwasawa algebra,

$$0 \longrightarrow M_1 \xrightarrow{\mu_{12}} M_2 \xrightarrow{\pi_{12}} F_1 \longrightarrow 0$$

$$0 \longrightarrow M_2 \xrightarrow{\mu_{23}} M_3 \xrightarrow{\pi_{23}} F_2 \longrightarrow 0$$

$$0 \longrightarrow M_1 \xrightarrow{\mu_{13}} M_3 \xrightarrow{\pi_{13}} F_1 \oplus F_2 \longrightarrow 0$$

Satisfying that

$$\mu_{13} = \mu_{23} \circ \mu_{12}, \quad \pi_{23} = (\pi_{13})_2, \quad \pi_{12} = (\pi_{13})_1 \circ \mu_{23} \tag{B.2}$$

where $(\pi_{13})_i$ for $i = 1, 2$ denotes the projection to the first or second component of $F_1 \oplus F_2$. Then

$$\phi_{M_3,M_1} = \phi_{M_3,M_2} \circ \phi_{M_2,M_1}$$

Proof. Let r_1, r_2 and r_3 be integers satisfying that $\operatorname{rank}_R F_2 = r_3 - r_2$ and $\operatorname{rank}_R F_1 = r_2 - r_1$. As in the construction of Proposition B.4.6, there are maps

$$\bigwedge^{r_3} M_3^+ \otimes \det(F_1^+) \otimes \det(F_2^+) \to \bigwedge^{r_2} M_2^+ \otimes \det(F_2^+) \to \bigwedge^{r_1} M_1$$

Using the canonical identification $\det(F_1 \oplus F_2) = \det(F_1) \otimes \det(F_2)$, the compatibility relations in (B.2) imply that the above composition coincides with the map

$$\bigwedge^{r_3} M_3^+ \otimes \det(F_1^+ \oplus F_2^+) \to \bigwedge^{r_1} M_1$$

Considering the dual maps, we deduce that $\phi_{M_3,M_1} = \phi_{M_3,M_2} \circ \phi_{M_2,M_1}$. □

B.4.3 Control of the Fitting ideals

For our purposes, the most important property of these maps is the control they make over Fitting ideals. The proof of this result depends on whether the coefficient ring is self-injective, a discrete valuation ring or the Iwasawa algebra.

Proposition B.4.12. If we have an exact sequence of R-modules over a self-injective ring R

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F \longrightarrow C \longrightarrow 0$$

where N and F are free R-modules of rank r and s, respectively. Let ψ be a generator of $\bigcap^r N$. Then the index $\Phi_{N,M}(\psi)$ (see Definition 3.1.4) is

$$\mathrm{ind}\big(\Phi_{N,M}(\psi)\big) = \mathrm{Fitt}^0_R(C)$$

Proof. Consider the composition

$$\bigwedge\nolimits^{r-s} N^+ \otimes \det(F^+) \longrightarrow \bigwedge\nolimits^{r-s} M^+ \otimes \det(F^+) \longrightarrow \bigwedge\nolimits^r N^+ \xrightarrow{\psi} R$$

where the first map is the one induced by $\mu^+ : N^+ \to M^+$, the second one is the dual of the rank reduction map constructed in Proposition B.4.6 and the last one is the map ψ. Note that the composition of the last two maps is $\Phi_{N,M}(\psi)$. Since the first map is clearly surjective, the image of this composition coincides with $\mathrm{ind}\big(\Phi_{N,M}(\psi)\big)$.

By Remark B.2.2, the Fitting ideal $\mathrm{Fitt}^0_R(C)$ is the image of the map

$$\mathrm{Fitt}^0_R(C) = \mathrm{Im}\left(\varepsilon : \bigwedge^s N \to \det(F)\right)$$

Since R is self-injective, this implies that

$$\det(F^+)\big[\mathrm{Fitt}^0_R(C)\big] = \ker\left(\det(F^+) \to \bigwedge^s N^+\right)$$

By the construction of the above maps, it implies that the image of the map

$$\bigwedge\nolimits^{r-s} N^+ \otimes \det(F^+) \xrightarrow{\beta} \bigwedge\nolimits^r N^+$$

is annihilated by $R\big[\mathrm{Fitt}^0_R(C)\big]$, since N free of rank r. By Proposition B.1.7, we can conclude that

$$\mathrm{ind}\big(\Phi_{N,M}(\psi)\big) = \mathrm{Fitt}^0_R(C) \qquad \square$$

We have an analogous control over Fitting ideals over discrete valuation rings.

Proposition B.4.13. If we have an exact sequence of R-modules over a discrete valuation ring R

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F \longrightarrow C \longrightarrow 0$$

where N and F are free R-modules of rank r and s, respectively. Let ψ be a generator of $\bigcap^r N$. Then the index $\Phi_{N,M}(\psi)$ (see Definition 3.1.4) is

$$\operatorname{ind}\big(\Phi_{N,M}(\psi)\big) = \operatorname{Fitt}^0_R(C)$$

Proof. We follow the structure of the proof of Proposition B.4.12. Similarly, $\operatorname{ind}\big(\Phi_{N,M}(\psi)\big)$ coincides with the image of the composition

$$\bigwedge\nolimits^{r-s} N^+ \otimes \det(F^+) \longrightarrow \bigwedge\nolimits^{r-s} M^+ \otimes \det(F^+) \longrightarrow \bigwedge\nolimits^{r} N^+ \xrightarrow{\psi} R$$

Similarly, by fixing an isomorphism $R \cong \det(F)$, the Fitting ideal $\operatorname{Fitt}^0_R(C)$ is the image of the map

$$\operatorname{Fitt}^0_R(C) = \operatorname{Im}\left(\varepsilon : \bigwedge^{s} N \to \det(F)\right)$$

Consider the image of the dual map

$$I := \operatorname{Im}\left(\det(F^+) \to \bigwedge^{s} N^+\right)$$

Every generator α of I has index $\operatorname{Fitt}^0_R(C)$, so there is an element $\beta \in \bigwedge^s N^+$ and a generator γ of $\operatorname{Fitt}^0_R(C)$ such that $\alpha = \gamma\beta$ and β projects to a non-zero element

$$\overline{\beta} \in \bigwedge^{s}(N^+ \otimes R/\mathfrak{m})$$

That implies that there exists a linearly independent set in the R/m vector space $N^+/\mathfrak{m}$, $S = \{e_1, \dots, e_s\}$, such that

$$\overline{\beta} = e_1 \wedge \cdots \wedge e_s$$

Consider a set $\{e_{s+1}, \dots, e_r\}$ extending S to a basis of $N^+/\mathfrak{m}$. Then

$$\alpha \wedge e_{s+1} \wedge \cdots \wedge e_r \in \bigwedge^{r} N^+$$

generates the ideal $\mathrm{Fitt}_R^0(C)\bigwedge^r N^+$. In addition, every element in

$$\bigwedge^{r-s} N^+ \otimes I \subset \bigwedge^{s} N^+$$

is clearly $\mathrm{Fitt}_R^0(C)$-divisible in the last exterior power. Hence,

$$\bigwedge^{r-s} N^+ \otimes I = \mathrm{Fitt}_R^0(C) \bigwedge^{s} N^+$$

Therefore,

$$\mathrm{ind}\big(\Phi_{N,M}(\psi)\big) = \mathrm{Fitt}_R^0(C) \qquad \square$$

When R is the Iwasawa algebra, the rank reduction map will no longer have full control over the Fitting ideals of the cokernel. However, in this case, we can proof that the index of $\Phi_{N,M}$ coincides with the Fitting ideal of C only up to finite index (see Notation 6.2.9)

Proposition B.4.14. If we have an exact sequence of Λ-modules over the Iwasawa algebra Λ,

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F \longrightarrow C \longrightarrow 0$$

where N and F are torsion-free Iwasawa modules of ranks r and s, respectively. Then $\bigcap^r N \cong \Lambda$.

If ψ is a generator of $\bigcap^r N$, the index $\Phi_{N,M}(\psi)$ is pseudo-equal to the Fitting ideal of C, as defined in Notation 6.2.9:

$$\mathrm{ind}\big(\Phi_{N,M}(\psi)\big) \approx \mathrm{Fitt}_\Lambda^0(C)$$

Proof. The structure theorem implies that N is pseudo-isomorphic to Λ^r. Then Lemma B.3.2 implies that $N^+ \cong \Lambda^r$, so

$$\bigcap^{r} N \cong \bigwedge^{r} \Lambda^r = \Lambda$$

We keep using the notation of the proofs of Propositions B.4.12 and B.4.13. After localising with a height 1 prime ideal β of Λ, we have that

$$\mathrm{Fitt}_\Lambda^0(C)_\beta = \mathrm{Im}\left(\bigwedge^{s} N_\beta \to \det(F)_\beta\right)$$

Note that N_β is a free Λ_β-module of rank r. Since Λ_β is a discrete valuation ring, the proof of Proposition B.4.13 implies that

$$\operatorname{ind}\big(\Phi_{N,M}(\psi)\big)_\beta \operatorname{Fitt}^0_\Lambda(C)_\beta = \operatorname{Fitt}^0_\Lambda(C)_\beta \bigwedge^{r+s} N^+_\beta$$

Since that holds for all height one prime ideals β, we conclude that

$$\operatorname{ind}\big(\Phi_{N,M}(\psi)\big) \approx \operatorname{Fitt}^0_R(C) \qquad \square$$

There is one special case in which we can prove that the index is contained in the Fitting ideal. That is because we can show that such index is a principal ideal, so it contains all ideals to which it is pseudo-equal.

Proposition B.4.15. Assume we have an exact sequence of finitely generated Iwasawa modules over the Iwasawa algebra Λ,

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} F \longrightarrow C \longrightarrow 0$$

where M, N and F are torsion-free Iwasawa modules of ranks

$$\operatorname{rank}_\Lambda N = r, \quad \operatorname{rank}_\Lambda F = S$$

for some natural numbers $r \geq s$. Let ψ be a generator of $\bigcap^r N$. Then $\operatorname{ind}\big(\Phi_{N,M}(\psi)\big)$ is the unique Fitting ideal containing $\operatorname{Fitt}^0_\Lambda(C)$ with finite index.

Proof. Since M is torsion-free, it is pseudo-isomorphic to a free Iwasawa module. Assume that its rank is larger than $r - s$. The additivity of the rank implies that C has positive rank, so

$$\operatorname{Fitt}^0_\Lambda(C) = 0$$

Since Λ contains no finite ideals, Proposition B.4.14 implies that

$$\operatorname{ind}(\Phi_{N,M})(\psi) = 0$$

Now assume that the rank of M is $r - s$. Then M is pseudo-isomorphic to Λ^{r-s}, so Lemma B.3.2 implies that

$$\bigcap^{r-s} M \cong \bigcap^{r-s} \Lambda^{r-s} = \Lambda$$

Then the index of every element in $\bigcap^{r-s} M$ is a principal ideal and, in particular, the index of $\phi_{N,M}(\psi)$. Since the principal ideals of the Iwasawa algebra contain all the ideals they are pseudo-equal to, we can conclude by Proposition B.4.14 that

$$\operatorname{Fitt}^0_\Lambda(C) \subset \operatorname{ind}\big(\Phi_{N,M}(\psi)\big) \qquad \square$$

B.4.4 Kernel of the rank reduction

When $s = 1$, it is possible to compute the kernel of the rank reduction maps.

Proposition B.4.16. Let M and N be modules over a ring R which is either self-injective, a discrete valuation ring or the Iwasawa algebra. In addition let

$$\phi_{N,M} : \bigcap^{r} N \to \bigcap^{r-1} M$$

be the rank reduction map induced by either Proposition B.4.6, B.4.7 or B.4.8, and an exact sequence

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\varepsilon} R$$

where F is a free Λ module of rank s. Consider also the map

$$\iota : \bigcap^{r} M \hookrightarrow \bigcap^{r} N$$

Then $\ker(\phi_{N,M}) = \mathrm{Im}(\iota)$.

Proof. We want to show that the following sequence is exact:

$$0 \longrightarrow \bigcap^{r} M \xrightarrow{\iota} \bigcap^{r} N \xrightarrow{\phi_{N,M}} \bigcap^{r-1} M \tag{B.3}$$

Assume first that R is either self-injective or a discrete valuation ring and consider the dual sequence:

$$\bigwedge^{r-1} M^{+} \longrightarrow \bigwedge^{r} N^{+} \longrightarrow \bigwedge^{r} M^{+} \longrightarrow 0 \tag{B.4}$$

The composite map is zero by construction. An element in the kernel of the second map can be expressed as a product $n_1 \wedge \cdots \wedge n_r$ where $n_r \in \ker(N^{+} \to M^{+})$, which belongs to the image of the first map. Hence this sequence is exact at $\bigwedge^{r-1} N^{+}$.

The sequence on the left follows from the surjectivity of μ^{+}. When R is self-injective, it follows from the exactness of the dual functor. When R is a discrete valuation ring, the cokernel of μ injects in $\mathrm{Ext}^1(\mathrm{Im}(\varepsilon), R)$, which vanishes since $\mathrm{Im}(\varepsilon)$ is a free module because it is contained in F.

Hence the sequence in (B.3) was also exact by the left exactness of the dual functor.

when R is the Iwasawa algebra, the sequence in (B.4) is no longer dual on the left, but the map

$$\bigwedge^r N^+ \to \bigwedge^r M^+$$

has finite cokernel. Then we can combine Lemma B.3.2 and the left exactness of the dual functor to conclude that the sequence in (B.3) is also exact. □

B.5 Base change

In this section, we establish some properties of the exterior biduals when changing the base ring R.

Assumption (BC). Let R be a self-injective ring, let I be an ideal of R such that $R[I]$ is a principal ideal, and let S be the quotient fitting in the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$$

Assume that S is also a self-injective ring.

Lemma B.5.1. Let M be an R-module such that $IM = 0$. Then there is a canonical isomorphisms

$$\mathrm{Hom}(M, R) = \mathrm{Hom}(M, S)$$

Proof. By the assumption on M, then

$$\mathrm{Hom}(M, R) = \mathrm{Hom}(M, R[I])$$

Since $R[I]$ is principal, it is canonically isomorphic to S. Then the lemma follows. □

Proposition B.5.2. Let M be an R-module such that $IM = 0$. Then there is a canonical isomorphism

$$\bigcap_R^r M = \bigcap_S^r M$$

Proof. Note that Lemma B.5.1 implies that

$$\mathrm{Hom}(M, R) = \mathrm{Hom}(M, S)$$

Since the first one is annihilated by I, we have that

$$\bigwedge_R^r \mathrm{Hom}(M, R) = \bigwedge_R^r \mathrm{Hom}(M, R) \otimes_R R/I = \bigwedge_S^r \mathrm{Hom}(M, S)$$

The proof can be completed by applying Lemma B.5.1 again. $\square$

Corollary B.5.3. Let M be an R-module. Then there is a canonical map

$$\bigcap_R^r M \to \bigcap_S^r M \otimes_R S$$

Proof. By the functoriality of the exterior bidual, there is a map

$$\bigcap_R^r M \to \bigcap_R^r M \otimes_R S$$

By Proposition B.5.2, there is an identification

$$\bigcap_R^r M \otimes_R S = \bigcap_S^r M \otimes_R S$$ $\square$

This identification commutes with the rank reduction map defined in Proposition B.4.6.

Proposition B.5.4. Let M, N be S-modules fitting in the exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow R$$

Since the image of the last map is contained in $R[I]$, we can consider the following exact sequence as well:

$$0 \longrightarrow M \longrightarrow N \longrightarrow R[I]$$

Under the identifications from Proposition B.5.2, the maps given by Proposition B.4.6

$$\bigcap^r N \to \bigcap^{r-s} M$$

are independent of the exact sequence.

Proof. It follows from the canonical identification $R[I]^* \cong S$ and tracking the constructions of the maps. $\square$

B.6 Exterior powers and ultraproducts

In this section, we show the commutativity properties between the maps between the exterior biduals defined in Proposition B.4.6 and the ultraproduct defined in Definition 4.1.13.

Proposition B.6.1. Let R be a self-injective, finite, local ring and let $(M_i)_{i\in\mathbb{N}}$ be a sequence finite R-modules whose orders are bounded above uniformly in i. Then, for any $r \geq 0$,

$$\mathcal{U}_i\left(\bigwedge^r M_i\right) = \bigwedge^r \mathcal{U}_i(M_i)$$

Proof. By the assumption on the orders of M_i, Lemma 4.1.21 implies that there is a set $S \in \mathcal{U}$ such that

$$\mathcal{U}_i(M_i) = M_j \ \forall j \in S$$

Then,

$$\bigwedge^r \mathcal{U}_i(M_i) = \bigwedge^r M_j \ \forall j \in S$$

Since the exterior powers of rank r of the index M_j are also uniformly bounded, Lemma 4.1.21 implies that

$$\bigwedge^r \mathcal{U}_i(M_i) = \mathcal{U}_i\left(\bigwedge^r M_i\right) \qquad \square$$

Corollary B.6.2. Let R be a self-injective, finite, local ring and let $(M_i)_{i\in\mathbb{N}}$ be a sequence finite R-modules whose orders are bounded above uniformly in i. Then, for any $r \geq 0$,

$$\mathcal{U}_i\left(\bigcap^r M_i\right) = \bigcap^r \mathcal{U}_i(M_i)$$

The same argument of Proposition B.6.1 can be used to prove the following modification.

Proposition B.6.3. Let R be a self-injective, finite, local ring and let $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ be a sequence finite R-modules whose orders are bounded above uniformly in i. Then,

$$\mathcal{U}_i\left(A_i \otimes B_i\right) = \mathcal{U}_i(A_i) \otimes \mathcal{U}_i(B_i)$$

Now we will show that the map in Proposition B.4.6 commutes with ultraproducts.

Proposition B.6.4. Let R be a self-injective ring and assume that, for all $i \in \mathbb{N}$, we have an exact sequence of R-modules,

$$0 \longrightarrow M_i \longrightarrow N_i \longrightarrow F_i$$

where F_i is a free R-module of rank s. Then Proposition B.4.6 induces, for all $i \in \mathbb{N}$, a map

$$\phi_i : \bigcap^{r+s} N_i \otimes \det(F_i) \to \bigcap^{r} M_i$$

By Corollary B.6.2 and Proposition B.6.3, the ultraproduct of this maps can be written as

$$\mathcal{U}_i(\phi_i) : \bigcap^{r+s} \mathcal{U}_i(N_i) \otimes \det\big(\mathcal{U}_i(F_i)\big) \to \bigcap^{r} \mathcal{U}_i(M_i)$$

Similarly, Proposition 4.1.20 induces an exact sequence,

$$0 \longrightarrow \mathcal{U}(M_i) \longrightarrow \mathcal{U}(N_i) \longrightarrow \mathcal{U}(F_i)$$

By Lemma 4.1.21, $\mathcal{U}(F_i)$ is a free R-module of rank s, so Proposition B.4.6 induces a map

$$\phi_{\mathcal{U}} : \bigcap^{r+s} \mathcal{U}_i(N_i) \otimes \det\big(\mathcal{U}_i(F_i)\big) \to \bigcap^{r} \mathcal{U}_i(M_i)$$

Then $\phi_{\mathcal{U}}$ coincides with $\mathcal{U}(\phi_i)$ as homomorphisms.

Proof. Consider the exact sequence

$$\mathcal{U}(F_i)^+ \longrightarrow \mathcal{U}(N_i)^+ \longrightarrow \mathcal{U}(M_i)^+ \longrightarrow 0$$

For any $r \in \mathbb{Z}_{\geq 0}$ can then construct the map

$$\bigwedge^{r} \mathcal{U}(M_i)^+ \otimes \det\big(\mathcal{U}(F_i)^+\big) \to \bigwedge^{r+s} \mathcal{U}(N_i)^+$$

By the identifications in Propositions B.6.1 and B.6.3, this map coincides with the ultraproduct of the maps

$$\mathcal{U}\left(\bigwedge^{r} M_i^+ \otimes \det\big(F_i^+\big) \to \bigwedge^{r+s} N_i^+\right)$$

Taking duals, we can check that

$$\phi_{\mathcal{U}} = \mathcal{U}_i(\phi_i)$$

$\square$

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