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INTRODUCCIÓN A LA FÍSICA DE AGUJEROS NEGROS

A BREAK COURSE IN BLACK HOLE PHYSICS

Supervisor: Diego Rubiera García

Alberto Angurel Andrés

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Singularity theorems which prove the existence of incomplete geodesics in some spacetimes.

Resumen:

La teoría de agujeros negros tiene gran importancia física, no sólo por el mero hecho de explicar el funcionamiento de unos objetos recientemente observados en la naturaleza, sino porque en ellos el espacio-tiempo tiene unas condiciones de curvatura mayores que las observadas un ningún otro lugar del universo. En este trabajo se estudian las singularidades que aparecen en los agujeros negros y en un universo en expansión, y se demuestran los teoremas de singularidad. Para ello, se discuten primero algunas propiedades que se imponen al espacio-tiempo: orientabilidad temporal, causalidad e hiperbolicidad global. Posteriormente, se trata la ecuación de Raychaudhuri y su relación con las condiciones de energía, que será utilizada para demostrar la existencia de singularidades. Por último, se muestran dichas singularidades en ciertos espacio-tiempos concretos. La existencia de singularidades, predicha por estos teoremas, supone la necesidad replantear la teoría gravitacional para explicar cómo se modifican las leyes físicas alrededor de ellas.

Palabras clave: agujero negro, singularidad, incompletitud geodésica, orientación temporal, condiciones de causalidad, dominios de dependencia, horizontes, Raychaudhuri, condiciones de energía.

Abstract:

The theory of black holes has a huge physical importance, not just to explain these systems that have been recently observed in Nature, but also because, inside them, spacetime attains stronger curvature conditions than the ones observed anywhere else in Nature. In this work, singularities appearing inside black holes and at the origin of an expanding Universe are studied and the singularity theorems are proven. To do that, some properties that are imposed to spacetime are discussed: time-orientability, causality and global hyperbolicity. Then, Raychaudhuri equation is exposed and it is discussed how it is related to energy conditions. It is used to prove the existence of singularities in some particular spacetimes. Finally, singularities are illustrated in some specific spacetimes. The existence of singularities, predicted by these theorems, supports the need for reconsidering the gravitational theory in order to explain how physical laws are modified near them.

Keywords: black hole, singularity, geodesic incompleteness, time-orientability, causality conditions, domains of dependence, horizons, Raychaudhuri, energy conditions.

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Introduction

Black holes are bounded regions of the spacetime in which gravity is so strong that nothing, not even light, can escape from it. Inside them, massive and non-massive particles would eventually reach the black hole center, which is a singularity of the spacetime. The aim of this work is to characterise these singularities and prove the theorems that claim their existence in a world described by General Relativity.

The first modern singularity theorem [1] was published in 1965, half a century after Einstein field equations, which are the base of General Relativity, were proposed. These theorems were the first important result afterwards which was not included in the original theory. They guarantee the existence of singularities in this theory under the assumption of several hypothesis which seem to be physically reasonable for black holes. First of all, they postulate a condition on spacetime, called the congruence condition, which could be related to the energy conditions satisfied by baryonic and leptonic matter and radiation. Then, they assume the existence of a trapped surface, which could be understood as one from which light cannot escape. This is a scenario that is found inside black holes. Finally, some theorems assume that spacetime is globally hyperbolic, which is a global condition imposed on it that assumes the existence of a surface which is a generalisation of an instant of time in the sense that every causal trajectory intersects it exactly once. However, this hypothesis could be avoided when working with bounded regions of the Universe, like the one inside a black hole.

The existence of singularities is a huge inconvenient when one is trying to model the physical world using the theory of General Relativity. In a singularity, spacetime disappears at some point, in a context that light or observers would eventually reach it and, consequently, leave the spacetime. Because classical theories should be deterministic, it is not physically reasonable that some observer could leave the spacetime. Then, another theory, different from General Relativity, is needed to explain how physical laws are modified near these singularities. Up to now, much research has been made to try explain it using different quantum gravity theories, although there is no accepted answer yet. A further discussion about that could be seen in [2].

The most known singularity takes place at the origin of the Universe in the ΛCDM cosmological model. It is commonly called *initial singularity* or *Big Bang*. Its particularity respect to other singularities happening inside black holes is that time is reversed. Instead of going to the singularity, light and massive particles comes from it. Then, General Relativity cannot explain what happened before Big Bang, not even which physical laws worked at that point.

As mentioned above, the main goal of this work is to make a complete proof of the singularity theorems published by Penrose. Before that, we discuss the different definitions of singularities in section 2. After that, time-orientability of a spacetime is defined in section 3, which means that future and past could be differentiated. In the same section, we impose the physical world not to have closed causal curves, which means that time travel is physically forbidden. In sections 4, 5 and 6, several physical and mathematical concepts are introduced, which will be necessary for proving the singularity theorems. In section 7, we introduce the Raychaudhuri equation, which has been sometimes considered as the first singularity theorem. It claims that every congruence of timelike or null geodesics contained in a spacetime satisfying the above mentioned congruence conditions would converge to a point within a finite amount of proper time, or affine parameter in case of null geodesics. Finally, proofs of the singularity theorems related to the existence of incomplete timelike and null geodesics are shown, respectively, in sections 8 and 9. Finally, in section 10, these theorems are illustrated in some metrics used to model black holes and the cosmological evolution. Along these work, natural units ($c = 1$) and the metric signature convention $(-, +, +, +)$ will be used.

1 Mathematical preliminaries

According to General Relativity, a spacetime $\mathcal{M}$ is a real, 4-dimensional, connected, Hausdorff differential manifold, in which a Lorentzian metric is defined. [1] Therefore, topological and geometrical techniques are needed for working with spacetime properties. Simple definitions and properties of topological spaces are assumed to be known. Nevertheless, it is interesting to mention that every spacetime $\mathcal{M}$ has to be paracompact, which means that every open cover $\{U_\alpha\}$ of $\mathcal{M}$ has a locally finite refinement $\{V_\beta\}$, what means that for every U_α, there is some V_β such that $V_\beta \subset U_\alpha$ and that every point has an open neighbourhood intersecting only finitely many V_β. Paracompactness of the spacetime is a key property for proving the existence of a limit for a sequence of curves. Moreover, some differential geometry, including the theory of geodesics, is assumed to be known too and can be reviewed, for instance, in [3], [4], [5], [6] or [7]. However, it is interesting to mention some results about the exponential map defined in a Riemannian or Lorentzian manifold and the existence of simple neighbourhoods around any point.

Definition 1.1. Let $\mathcal{M}$ be a Riemannian or a Lorentzian manifold and let v be a tangent vector at some point $p \in \mathcal{M}$. The exponential map of v, denoted by $\exp(v)$, is the point $\gamma_v(1)$, where γ_v is the unique geodesic satisfying that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Of course, this definition

[1] Rigorously, the spacetime should be denoted by $(\mathcal{M}, g_{\mu\nu})$, where $g_{\mu\nu}$ is the metric tensor defined on it. However, when there is no risk of misunderstanding, it will be referred as $\mathcal{M}$.

applies only to tangent vectors $v \in T_p\mathcal{M}$ such that γ_v could be defined that far. Using the inverse function theorem, it is proven in [3] and [4] that the exponential map is smooth and defines a diffeomorphism from an open neighbourhood of the null vector in the tangent space of p onto an open neighbourhood of p in $\mathcal{M}$. A neighbourhood of p such that the exponential map defines a diffeomorohism onto it is called a *normal neighbourhood* of p.

Definition 1.2. The exponential map is defined similarly in tangent vectors which are orthogonal to a submanifold. It can be seen in [4] that it defines a diffeomorphism onto an open neighbourhood of the manifold.

Definition/Proposition 1.3. A *simple set* is a subset of $\mathcal{M}$ which is a normal neighbourhood for every one of its points. It is proven in [3] and [4] that every point has a simple neighbourhood and that spacetime can be covered by a locally finite collection of simple sets. In particular, it implies that any compact set can be covered by a finite number of them.

Simple and normal neighbourhoods are important for proving local properties of spacetimes because normal coordinates could be defined on them. Being U a normal neighbourhood around p, there is a set of coordinates (t, x, y, z) such that p is represented by $(0, 0, 0, 0)$, $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and geodesics through p can be written as $(t_0\lambda, x_0\lambda, y_0\lambda, z_0\lambda)$, where λ is an affine parameter.

The most important property of simple neighbourhoods is the following one.

Proposition 1.4. If N is a simple region, any two points $p, q \in \bar{N}$ can be connected by a unique geodesic in N, denoted by pq. Moreover, its length, which will be defined in section 5, is a continuous function of its endpoints.

2 Definition of singularity

Singularities in a spacetime $\mathcal{M}$ could be understood as scenarios in which a General Relativity breakdown happen. It is reasonable to think that this breakdown only happens in the theory, not in the physical world. Then, an updated theory is needed to explain the physical behaviour at the singularities and maybe a more advanced quantum physics could do that. Different examples of these situations could be found in Friedmann-Lemâitre-Robertson-Walker metric (Big Bang, at $t = 0$) or in Schwarzschild metric (at point $r = 0$). However, there is no observational evidence from physical laws at singularities because information inside an Schwarzschild black hole cannot leave it and temperature was so high during Big Bang that photons could not travel long distances without interacting with other particles, so the Universe was opaque. At this point, it is interesting to mention that we are interested in singularities which are intrinsic of the spacetime, not due to the result of using a special coordinate system. For instance, Schwarzschild metric has a divergence at $r = r_S = 2m$ which could be avoided using Kruskal coordinates.

However, there are some conceptual difficulties related to how one could define a singularity. Historically, some different definitions have been used, even tough the most accepted one nowadays is related to *geodesic incompleteness*. Geodesic incompleteness in a spacetime is the existence of geodesics which are past or future inextendible but their affine parameter is only defined within a bounded interval. Those geodesics are called *incomplete*. Assuming that a geodesic is future incomplete, an observer moving along that geodesic would complete its movement in a finite amount of proper time, while General Relativity could not explain what would happen after that, failing to be a deterministic theory.

There has been some discussion about how correct is to restrict the definition of singular spacetimes to geodesic incompleteness, instead of considering a wider variety of paths. For instance, we can consider paths such that their total acceleration is bounded. These paths may represent travels made by an observer having a finite amount of 'fuel' for modifying its trajectory. In [8], Geroch constructed a geodesically complete spacetime containing an incomplete causal path with bonded acceleration, although he classified it as non-singular. Even though there is no agreement about how to define singularities, we will understood by a singular spacetime as one that is geodesically incomplete.

In this context, singularities are related to 'holes' in spacetime. A problem that arises from this definition is that a point $p \in \mathcal{M}$ could be removed artificially and every geodesic containing p would become incomplete, so spacetime would be singular. To avoid this situation, it is necessary to work with intextendible spacetimes, which means that spacetime is not isometric to a proper subset of another spacetime. This notion ensures that singularities are due to a physical reason and not related to the fact that the spacetime considered is just not big enough. A further discussion related to the different ways one could define singular spacetimes appears in [9].

3 Orientability and causality

Let $\mathcal{M}$ be a Lorentzian spacetime. Then, the tangent space at every point $p \in \mathcal{M}$ is isomorphic to Minkowski space. Therefore, timelike vectors in this tangent space can be divided in two connected components. A spacetime is called *time-orientable* if there exists a continuous designation about which component is the future one as p varies over $\mathcal{M}$. Provided that spacetime is time-orientable, there are several physical processes that could differentiate future from past. One of them could be the second law of thermodynamics, that says that future is the time direction in which the total entropy of the Universe increases. In the following, we will assume that spacetime is time-orientable and such future designation has been made.

Causality is related to the possibility that some point in spacetime could be influenced by another one, meaning that there exists a causal curve joining these two points, so information could travel from one point to the other. In this context, we understand a *curve* as a continuous and piecewise smooth map $\gamma : I \to \mathcal{M}$, where $I \subset \mathbb{R}$ is an interval. It is said to be *causal* (resp. *timelike*) if its tangent vector is causal (resp. timelike) at every point in which it is defined. Each interval $I' \subset I$ such that its restriction of γ is regular is called a *segment.* Then, causality is established in the following definition.

Definition 3.1. The causality relations in a space-time $\mathcal{M}$ are defined as follows:

- $p \ll q$ if there is a future directed timelike curve from p to q.
- $p < q$ if there is a future directed causal curve from p to q.

The different future sets of a point are defined as the sets of points which can be reached by a timelike (resp. causal) curve:

- Chronological future: $I^+(p) = \{q \in M : p \ll q\}$.
- Causal future: $J^+(p) = \{q \in M : p \leq q\}$

The chronological and causal pasts, $I^-(p)$ and $J^-(p)$, are defined similarly.

Given a set $S \subset \mathcal{M}$, its chronological (resp. causal) future set is the union of the chronological (resp. causal) futures of every point $p \in S$. Again, the chronological and causal pasts of a set S are defined similarly.

The chronological and causal futures satisfy some interesting properties.

Proposition 3.2. $I^+(p)$ is open for every $p \in \mathcal{M}$.

Proof. Let $q \in I^+(p)$ and let γ be a future directed timelike curve from p to q. Let U be a simple neighbourhood of q and $r \in U$ be a point in γ slightly before q. Then, $I^+(r,U)$ is diffeomorphic to an open set contained in the future-pointing vectors in the tangent space at r, so $I^+(r,U)$ is open. As $q \in I^+(r,U) \subset I^+(p)$ is arbitrary, $I^+(p)$ is open. □

Corollary 3.3. Given $S \subset \mathcal{M}$, $I^+(S)$ is an open subset of $\mathcal{M}$.

Proposition 3.4. Given a subset $S \subset \mathcal{M}$, then $I^+(S) = I^+(\overline{S})$.

Proof. Clearly, $I^+(S) \subset I^+(\overline{S})$. Conversely, if $q \in I^+(\overline{S})$, then $I^-(q) \cap \overline{S} \neq \emptyset$, so $I^-(q) \cap S \neq \emptyset$ because $I^-(q)$ is open by proposition 3.2. Then $q \in I^+(S)$. □

Proposition 3.5. Given points $p, q, r \in \mathcal{M}$, if $p \ll q$ and $q < r$, or $p < q$ and $q \ll r$, then $p \ll r$.

Proof. First, we are proving it when p, q and r all lie in the closure of a single simple neighbourhood. For that, define $\phi(x,y) := -g\left(\exp_x^{-1}(y), \exp_x^{-1}(y)\right)$ as the square of the length of the geodesic segment xy. Clearly $\phi(a,x)$, for a fixed a, increases when $x > a$ moves along a future-pointing timelike curve or along a future directed causal curve whose tangent vector is not proportional to the one of the geodesic ax. Then, this proposition is clear when all p, q, r lie on a single simple neighbourhood. A more detailed proof of this statement could be seen in [10].

Without loss of generality, we can assume that $p \ll q$ and $q < r$. Then, there is a timelike curve γ from p to q and a causal curve λ from q to r. As λ, considered as its image set, is compact, it can be covered by a finite number of simple regions $N_1, \dots, N_m$. Set $x_0 = q \in N_{i_0}$ and let x_1 be the future endpoint of the connected component of $\lambda \cap \overline{N_{i_0}}$. Choose $y_1 \in N_{i_0}$ in the final segment of γ, but distinct from x_0. As they are contained in the closure of the simple region N_{i_0}, it is possible to construct the geodesic segment $y_1 x_1$, which is timelilke because of the first part of this proof. At this point, either $x_1 = r$ or $x_1 \notin N_{i_0}$, so there exists some different i_1 such that $x_1 \in N_{i_1}$. Then, repeating this argument a finite number of times, it is possible to construct a timelike curve form p to r, so $p \ll r$. □

Corollary 3.6. If $q \in J^+(p) \setminus I^+(p)$, then there exists a null geodesic from p to q.

Proof. As $q \in J^+(p)$, there exists a causal curve γ from p to q. Because γ is compact, it can be covered by finitely many normal neighbourhoods $N_{i_1}, \dots N_{i_r}$ (possibly repeated) such that there exist $x_i \in N_i \cap N_{i+1} \cap \gamma$. Then, $x_i, x_{i+1} \in N_{i+1}$ and $x_i x_{i+1}$ is a null geodesic because, otherwise, proposition 3.5 would imply that $q \in I^+(p)$. Therefore, there is a broken null geodesic from p to q.[2] However, choosing a simple neighbourhood N_i' of x_i, and denoting by γ_i^- and γ_i^+ the incoming and outgoing geodesic segments, respectively, if $\gamma_i^- \cup \gamma_i^+$ is not a geodesic segment, it would be possible to choose $\alpha \in \gamma_i^- \cap N_i$ and $\beta \in \gamma_i^+ \cap N_i$ such that $\alpha\beta$ is a timelike geodesic (reasoning similarly to the first part of the proof of proposition 3.5). In that case, $q \in I^+(p)$ due to what was commented above. By repeating this argument a finite amount of times, we get that γ is a null geodesic. □

Lemma 3.7. Given a subset $S \subset \mathcal{M}$, then $J^+(S) \subset \overline{I^+(S)}$. The equality holds whenever $J^+(S)$ is closed.

Proof. Only is it necessary to be proven when $S = \{p\}$. Obviously, $p \in \overline{I^+(p)}$, so let $q > p$ and let γ be a future directed causal curve from p to q. Let U be a simple neighbourhood of q and $q' \in U$ be a point in γ slightly before q. Then $q \in J^+(q',U) = \overline{I^+(q',U)}$ because of the properties of simple neighbourhoods. However, $I^+(q',U) \subset I^+\left(J^+(p)\right) = I^+(p)$, by proposition 3.5. Hence, $q \in \overline{I^+(p)}$. □

[2]By a broken geodesic, we understand a curve γ in which there is a finite sequence $\{x_1, \dots, x_n\}$ such that every $\gamma|_{[x_i, x_{i+1}]}$ is a geodesic.

It is technically possible for a Lorentzian manifold to admit a closed causal curve. Physically, it is not acceptable because a particle would be able to come back to the moment it was created and, perhaps, impede that process. In other words, time travels would be physically allowed in this scenario, what is unacceptable. Moreover, neither is accepted the fact that a causal curve could come arbitrarily close to intersect itself. In that case, arbitrary small perturbations of the metric could produce a closed causal curve. Therefore, it is usually imposed the *strong causality condition* on the spacetime: for every $p \in M$ and every neighbourhood U of p, there exists another neighbourhood V of p, contained in U, such that no causal curve intersects V more than once.

4 Domains of dependence and horizons

In a spacetime $\mathcal{M}$, there is a kind of subsets of special importance, which are defined as follows.

Definition 4.1. A subset S of a spacetime $\mathcal{M}$ is called *achronal* if every two points $x, y \in S$ are not chronologically related, which means that $x \not\ll y$ and $y \not\ll x$.

It is necessary to introduce two technical definitions before defining the domains of dependence.

Definition 4.2. Let S be a closed achronal set. The *edge* of S consists of the points $p \in S$ such that every neighbourhood U of p contains points $q_1 \in I^-(p, U)$ and $q_2 \in I^+(p, U)$ and a causal curve from q_1 to q_2 which does not intersect S.

Definition 4.3. Let $\gamma : I \to \mathcal{M}$ be a curve in a spacetime. A point $p \in \mathcal{M}$ is said to be a *future endpoint* of γ if, given any neighbourhood U of p, there exists $t_0 \in I$ such that $\gamma(t) \in U$ $\forall t \geq t_0$.[3] A curve is *future-inextendible* if it has no future-endpoint. Past endpoints and past-inextendibility are defined similarly.

Next lemma will let us to derive some properties about spacetimes which satisfy the strong causality condition.

Lemma 4.4. Let $\mathcal{M}$ be a strongly causal spacetime and let $K \subset \mathcal{M}$ be a compact subset. Then, every causal curve γ confined within K must have both past and future endpoints in K.

Proof. Without loss of generality, we may assume that the curve parameter is defined in the whole set $\mathbb{R}$. Let $\{t_i\}$ be a sequence diverging to infinity and let $p_i := \gamma(t_i)$. The compactness of K implies the existence of an accumulation point $p \in K$. Hence, for every neighbourhood U of p, infinitely many p_i belong to U. If p would not be a future endpoint of γ, γ would never remain in U, contradicting the strong causality condition at p. Similarly, γ has a past endpoint in K. $\square$

Definition 4.5. For every subset S of a spacetime $\mathcal{M}$, we define the *future domain of dependence*, $D^+(S)$, as the set of points $p \in \mathcal{M}$ such that every past-inextendible causal curve containing p intersects S.[4] Similarly, the *past domain of dependence*, $D^-(S)$, is the set of points $p \in M$ such that every future-inextendible causal curve through p intersects S. The full domain of dependence is defined as $D(S) := D^+(S) \cup D^-(S)$ and could be understood as the set of points $p \in \mathcal{M}$ such that every future and past-inextendible causal curve through p intersects S.

Lemma 4.6. Given a closed achronal set S, $\overline{D^+(S)}$ is the set T of points $p \in \mathcal{M}$ such that every past-pointing inextendible timelike curve through p meets S.

[3]This definition is the mathematical condition for the fact that the curve cannot be defined in a strictly bigger interval of the affine parameter. If that was the case, it would exists the above mentioned limit. Conversely, in case that limit exists, the curve could be defined in a semi-closed interval by adding this limit and later could be extended.

[4]Physically, the future domain of dependence could be understood as the set of points which are completely determined by the set S.

Proof. On the one hand, we will see that $\overline{D^+(S)} \subset T$. Suppose by contradiction that there exists $p \in \overline{D^+(S)} \setminus T$. Then, there is a past-inextendible timelike curve α starting at p which does not meet S. Hence, $p \notin S$, so there is a causally convex neighbourhood U of p which does not intersect S. Choosing a point $r \in U$ slightly over p in the past direction of α, then $I^+(r, U)$ contains p, so it contains some $q \in D^+(S)$ too. The curve obtained by juxtaposition of the geodesic segment from q to r in U and the segment of α past r constitutes a past-inextendible timelike curve which does not meet S, contrary to $q \in D^+(S)$.

On the other hand, we will show that $T \subset \overline{D^+(S)}$. Given $q \notin \overline{D^+(S)}$, consider $M \setminus \overline{D^+(S)}$ as a manifold and choose $r \in I^-\left(q, M \setminus \overline{D^+(S)}\right)$. Then, there is a past-pointing timelike curve γ from q to r and a past-inextendible causal λ curve starting from r which misses S. Juxtaposing them and transforming the resulting curve in a timelike one like the proof of proposition 3.5, we get that $q \notin T$. □

Definition 4.7. A *Cauchy surface* Σ in a spacetime $\mathcal{M}$ is a closed achronal set satisfying that $D(\Sigma) = \mathcal{M}$.[5] A spacetime $\mathcal{M}$ is said to be *globally hyperbolic* if it contains a Cauchy surface.[6]

Proposition 4.8. Given a Cauchy surface Σ, every inextendible causal curve γ intersects Σ, $I^+(\Sigma)$ and $I^-(\Sigma)$.

Proof. By definition 4.7, there is $a \in I$ (being I the interval in which γ is defined) such that $\gamma(a) \in \Sigma$. As γ is inextendible, it has no endpoints, so it could be extended to an open neighbourhood I' of a. In that case, there would be $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset I'$. Then, $\gamma(a + \varepsilon/2) \in I^+(\Sigma)$ and $\gamma(a - \varepsilon/2) \in I^-(\Sigma)$, assuming that γ is future directed. □

We need to define some kind of limit for a sequence of curves in order to derive some properties of globally hyperbolic spacetimes.

Definition 4.9. Let $\{\gamma_n\}$ be a sequence of future directed causal curves in a spacetime $\mathcal{M}$ and let $\mathcal{R}$ be a convex covering of $\mathcal{M}$. A *limit sequence* for $\{\gamma_n\}$ relative to $\mathcal{R}$ is a (possibly finite) sequence $\{p_0, p_1, \ldots\} \subset \mathcal{M}$ such that:

1. For each p_i, there is a subsequence $\{\gamma_m\}$ and numbers $s_{m_0} < s_{m1} < \ldots < s_{mi}$ such that
   (a) $\lim_{m\to\infty} \gamma_m(s_{m_j}) = p_j \ \forall j \leq i$.
   (b) For every $j < i$, the points p_j, p_{j+1} and the segments $\gamma_m|_{[s_{mj}, s_{m,j+1}]}$ are all contained in a single $C_j \in \mathcal{R}$.
2. If $\{p_i\}$ is infinite, it is non-convergent while, in case it is finite, it consists of more than one point and there is no strictly longer sequence satisfying the above mentioned conditions.

Lemma 4.10. Let $\{\gamma_n\}$ be a sequence of future-pointing causal curves satisfying that $\{\gamma_n(0)\} \to p$ and such that there exists a neighbourhood of p containing only finitely many curves γ_n. Then γ_n has a limit sequence starting at p relative to any convex covering $\mathcal{R}$.

Proof. Since $\mathcal{M}$ is paracompact, it has a locally finite subcovering $\mathcal{R}'$ formed by open sets U_α such that every $\overline{U}_\alpha$ is compact, causally convex, satisfies the strong causality condition and is contained in some member of $\mathcal{R}$. By hypothesis, we can suppose that $\mathcal{R}'$ contains some U_0 such that infinitely

[5] A Cauchy surface is a generalization of an instant of time in Minkowski spacetime.

[6] Cauchy surfaces have huge importance in General Relativity, specially in the Cauchy problem. If a Cauchy surface is a smooth hypersurface, a metric tensor defined on it could be extended uniquely to the whole spacetime. See [6] for further discussion.

many γ_n start in U_0 but leave $\overline{U_0}$ eventually. Let $\gamma_n(s_{n1})$ be the first point of γ_n in δU_0. Passing to a further subsequence, by the compactness of δU_0, $\{\gamma_n(s_{n1})\}$ converges to a point $p_1 \in \delta U_0$.

Now choose some $U_1 \in \mathcal{R}'$ containing p_1. If infinitely many of the remaining γ_n leave $\overline{U_1}$, same argument constructs the point $p_2 \in \delta U_1$. We repeat this argument as many times as possible, choosing the element of $\mathcal{R}'$ that has been used less times when there is more than one candidate available. Clearly, the first condition in definition 4.9 is satisfied.

Suppose that the sequence obtained $\{p_i\}$ is infinite and convergent to some $q \in \mathcal{M}$. For some $U \in \mathcal{R}'$ containing q, $p_i \in U$ for all but a finite number of them. Since $\overline{U}$ is compact and $\mathcal{R}'$ is locally finite, only finitely many members of $\mathcal{R}'$ intersect U. Moreover, U could have been chosen only a finite number of times because, when it is chosen, next point does not belong to U. However, some member of $\mathcal{R}'$ intersecting with U should have been chosen infinitely many times, by the pigeonhole argument, and this violates the election order used.

Suppose now that the resulting sequence is finite: $p_0 < \ldots < p_k$. Since the construction cannot be continued, only finitely many of the remaining γ_n leave $\overline{U_k}$. The γ_n trapped in $\overline{U_k}$ are extendible by lemma 4.4, so we can suppose they are defined in an interval $[0, \xi_m]$. By compactness, a further subsequence of $\gamma_m(\xi_m)$ converges to some $q \in \overline{U_k}$. If $q = p_k$, the subsequence cannot be extended while, if $q \neq p_k$, the sequence $p_0 < \ldots < p_k < q$ is an inextendible limit sequence of $\{\gamma_n\}$. □

Definition 4.11. Given a sequence of curves $\{\gamma_n\}$, joining points p_i and p_{i+1} of a limit sequence by the geodesic segment contained in C_i gives a broken geodesic called *quasi-limit* of $\{\gamma_n\}$ with vertices $\{p_i\}$. If the limit sequence is infinite, then the quasi-limit is future-inextendible. This is the case when each γ_n is future-inextendible.

Proposition 4.12. Every globally hyperbolic spacetime $\mathcal{M}$ satisfies the strong causality condition.

Proof. For being $\mathcal{M}$ globally hyperbolic, it is satisfied that $\mathcal{M} = I^+(\Sigma) \cup \Sigma \cup I^-(\Sigma)$, where Σ is a Cauchy surface. Suppose, for the sake of contradiction, that the strong causality condition is violated at some point $p \in \mathcal{M}$. Then, there is a sequence of curves γ_n defined on $[0, 1]$ satisfying that both $\{\gamma_n(0)\}$ and $\{\gamma_n(1)\}$ converge to p, but every γ_n leaves some fixed neighbourhood of p. If the limit sequence given by lemma 4.10 is finite, the correspondent quasi-limit λ would be a closed timelike curve. Extending it by going through it again infinitely many times would produce an inextendible timelike curve which would intersect Σ more than once or would not do it at any point, contrary to definition 4.7.

On the other hand, if the limit sequence is infinite, the corresponding quasi-limit is future inextendible, so proposition 4.8 implies that it intersects $I^+(\Sigma)$, so some $p_i \in I^+(\Sigma)$. By reparametrization, we can suppose that there exists $s \in (0, 1)$ such that $\lim_{m\to\infty} \gamma_m(s) = p_i$, where $\{\gamma_m\}$ is a subsequence of $\{\gamma_n\}$. Using lemma 4.10 dually, we get a past directed limit sequence for $\gamma_m|_{[s,1]}$ starting at p. If it is finite, it should end at p_i and then $p_i < p$ and $p > p_i$, existing a closed causal curve, which is the contradiction mentioned before. If it is infinite, the resulting quasi-limit is a past-inextendible causal curve, so it must intersect $I^-(\Sigma)$ and then infinitely many $\gamma_m|_{[s,1]}$ would do that. Since $\gamma_m(s) \in I^+(\Sigma)$ for infinitely many m and γ_m are future directed, it contradicts the achronality of Σ. Therefore, the strong causality condition must be satisfied in $\mathcal{M}$.

□

Theorem 4.13. Given any two causally related points $p \leq q$ in a globally hyperbolic spacetime $\mathcal{M}$, the set $J(p, q) := J^+(p) \cap J^-(q)$ is compact.

Proof. If $p = q$, the strong causality condition given by proposition 4.12 implies that $J(p, p) = \{p\}$, which is a compact set. Suppose, otherwise, that $p < q$. Because the topology of $\mathcal{M}$ is second countable, given a sequence $\{x_n\} \subset J(p, q)$, we just need to show that it has an accumulation point.

Let γ_n be future directed causal curves from p to q through x_n. Let $\mathcal{R}$ be a covering of $\mathcal{M}$ by convex open simple sets C such that $\overline{C}$ is compact and contained in a convex open set. Lemma 4.10 guarantees the existence of a limit sequence starting at p. If it is finite, it means that some $p_k = q$ and any subsequence $\{\gamma_m\}$ as the one in definition 4.9 satisfies, by the pigeonhole argument, that there is an $i < k$ such that $x_m \in \gamma_m|_{[s_{mi}, s_{m,i+1}]}$ for infinitely many m. Passing to this subsequence, the points x_m lie in a single member C of $\mathcal{R}$. By the properties of simple sets like C, $\{x_m\}$ has an accumulation point $x \in \overline{C}$ such that $p_i \leq x \leq p_{i+1}$. Hence $x \in J(p, q)$.

We just need to get a contradiction when $\{p_i\}$ is infinite. In that case, the respective quasi-limit would be future-inextendible, so it would intersect $I^+(\Sigma)$. Reparametrizing, there is a subsequence $\{\gamma_m\}$ and $s \in (0,1)$ such that $\gamma_m(s)$ converges to a vertex $p_i \in I^+(\Sigma)$. Since $p_i \neq q$, dual of lemma 4.10 gives a past-directed limit sequence $\{q_i\}$ for the curves $\gamma_m|_{[s,1]}$. If $\{q_i\}$ were finite, then $\{p_0, p_1, \ldots, p_i, \ldots, q_1, q\}$ is a finite limit sequence of $\{\gamma_n\}$, so the preceding case works. Otherwise, when $\{q_i\}$ is infinite, the new quasi-limit $\overline{\lambda}$ is a past inextendible causal curve, so it reaches $I^-(\Sigma)$. Thus, some $\gamma_m|_{[s,1]}$ does it. As $\gamma_m(s) \in I^+(\Sigma)$, this fact contradicts the achronality of Σ. □

Now we turn our attention into the boundary of this region, which will be a key concept for proving the singularity theorems.

Definition 4.14. The *future Cauchy horizon* of an achronal closed set S is defined as

$$H^+(S) = \left\{x \in \overline{D^+(S)} : D^+(S) \cap I^+(x) = \emptyset\right\} = \overline{D^+(S)} \setminus I^-\left(D^+(S)\right) = \overline{D^+(S)} \setminus I^-\left(\overline{D^+(S)}\right)$$

Similarly, the *past Cauchy horizon* is defined and the total *Cauchy horizon* is $H(S) = H^+(S) \cup H^-(S)$.

Lemma 4.15. If S is a closed achronal set, then $\delta D^+(S) = S \cup H^+(S)$.

Proof. By the achronality of S, we have the inclusion $S \cup H^+(S) \subset \delta D^+(S)$. Conversely, suppose there is some $p \in \delta D^+(S) \setminus (S \cup H^+(S))$, then $p \in \overline{D^+(S)} \setminus S$ and, by lemma 4.6, $p \in I^+(S)$. Furthermore, $p \in \overline{D^+(S)} \setminus H^+(S)$, so there exists $q \in I^+(p) \cap D^+(S)$. Thus, $I^+(S) \cap I^-(q)$ is an open neighbourhood of p which is contained in $D^+(S)$ for being S achronal. This fact contradicts that $p \in \delta D^+(S)$. □

Theorem 4.16. Given an achronal set S, every point $p \in H^+(S)$ lies on a null geodesic γ contained entirely within $H^+(S)$ which is either past inextendible or has a past endpoint on $edge(S)$.

Proof. Let $p \in H^+(S) \setminus edge(S)$. Then either $p \in I^+(S)$ or $p \in S$ due to lemma 4.6.

In the first case, since $p \notin I^-(D^+(S))$ because of definition 4.14, there is a past directed inextendible timelike curve starting from every $q \in I^+(p)$ which does not intersect S. Let $\{q_n\} \subset I^+(p)$ be a sequence converging to p and $\{\gamma_n\}$ its corresponding sequence of curves. Then, by lemma 4.10 and definition 4.11, there is an inextendible quasi-limit γ through p which does not enter the open set $I^+(S) \cap I^-\left(D^+(S)\right) \subset D^+(S)$ because no γ_n does that. Since $I^-(p) \subset I^-(\overline{D^+(S)}) = I^-(D^+(S))$, γ is a past directed causal curve which does not enter $I^-(p) \cap I^+(S)$, so $\gamma \cap I^+(S)$ has to be a null geodesic. Moreover, $\gamma \cap I^+(S) \subset \overline{D^+(S)}$ because, otherwise, proposition 3.5 and lemma 4.6 would construct a past-inextendible timelike curve which skips S, contradicting this lemma 4.6. Thus, $\gamma \cap I^+(S) \subset H^+(S)$, so there is a past-pointing null geodesic segment contained in $H^+(S)$.

In the second case, $p \in S \setminus edge(S)$, so there exists a neighbourhood U of p such that no causal curve starting from a point $q \in I^+(p) \cap U$ can intersect $I^-(p) \cap U$ without meeting S. A similar argument gives the existence of a past-pointing geodesic through p, which will be a null geodesic at p because the quasi-limit γ will not intersect $I^-(p) \cap U$. Furthermore, $\gamma \cap U$ will not intersect $I^-(S)$ because no γ_n do that, so it will remain in S. Similarly, it will not intersect $I^-(D^+(S))$ either, so it will remain in $H^+(S)$

In both cases, if a past-pointing null geodesic γ leaves $H^+(S)$, the segment contained in the horizon has a past endpoint $r \in H^+(S)$ because $H^+(S)$ is closed. If $r \notin edge(S)$, there would be another geodesic segment γ' starting from r and contained in $H^+(S)$. Furthermore, by the definition of r, γ' cannot be the continuation of γ, so there is a timelike curve from a point in γ to a point in γ', violating the achronality of $H^+(S)$. □

5 Time separation

Definition 5.1. Let $\gamma : [a, b] \to \mathcal{M}$ be a piecewise regular causal curve. Its length is defined as:

$$L(\gamma) = \int_a^b \sqrt{-\dot{\gamma}_\mu(t)\dot{\gamma}^\mu(t)}\, dt$$

Definition 5.2. For every $p, q \in \mathcal{M}$, their *time separation* is defined as

$$\tau(p, q) = \sup\{L(\gamma) : \gamma \text{ is a timelike curve from } p \text{ to } q\}$$

where $\tau(p, q) := \infty$ if the set of legths is unbouded and $\tau(p, q) := 0$ if it is empty.

Proposition 5.3. Let $N \subset \mathcal{M}$ be a simple region and let $p, q \in N$ such that pq is future causal. Then $L(pq) = \tau_N(p, q)$. [7]

Proof. If pq is null, $q \in J^+(p) \setminus I^+(p)$, so the result is clear. Otherwise, choose Minkowski normal coordinates with origin at some point $r \in I^-(p)$ lying along the past extension of pq and define, in an open neighbourhood of pq, a new set of coordinates given by

$$T = \sqrt{t^2 - x^2 - y^2 - z^2}, \quad X^1 = \tfrac{x}{t}, \quad X^2 = \tfrac{y}{t}, \quad X^3 = \tfrac{z}{t}$$

This metric takes the form $ds^2 = -dT^2 + h_{ab}X^aX^b$, where h_{ab} is a positive definite 3 dimensional metric. In this metric, curves $(X_1, X_2, X_3) = (k_1, k_2, k_3)$ are geodesics. Let α be another curve from p to q. Then

$$L(\alpha) = \int_{T_p}^{T_q} \sqrt{1 - h_{ab}\frac{dX^a}{dT}\frac{dX^b}{dT}}\, dT$$

from which it is clear that the maximum takes place whenever X_1, X_2 and X_3 are constant. That means that geodesics locally maximize proper length. □

Lemma 5.4. The time separation function $\tau : \mathcal{M} \times \mathcal{M} \to [0, \infty] : \ (p, q) \mapsto \tau(p, q)$ is lower semicontinuous.

Proof. Let $p, q \in M$. Then, for every $\delta > 0$, we must find neighbourhoods U and V of p and q, respectively, such that $\tau(p', q') > \tau(p, q) - \delta$ for every $p' \in U$ and $q' \in V$.

If $\tau(p, q) = 0$, there is nothing to prove. Otherwise, there is a timelike curve α from p to q such that $L(\alpha) > \tau(p, q) - \delta/3$. Let N be a simple region of q and let q_1 be a point of α in N. If q_1q is the geodesic segment in N from q_1 to q, its length depends continuously on its endpoints due to proposition 1.4. Therefore, there exists a neighbourhood V of q such that for every $q' \in V$, $L(q_1q') > L(q_1q) - \delta/3 \geq L(\alpha|_{[q_1,q]}) - \delta/3$, where the last inequality comes from proposition 5.3.

Same construction gives a neighbourhood U around p such that every $p' \in U$ and every $q' \in V$ can be joined in an obvious way by a causal curve β satisfying that

$$L(\beta) > L(\alpha) - 2\delta/3 > \tau(p, q) - \delta \Rightarrow \tau(p', q') > \tau(p, q) - \delta$$

□

[7] Here, the subindex remarks that N is being considered as whole spacetime, so paths eventually leaving N are not taken into consideration.

Lemma 5.5. Suppose that the strong causality condition holds on a compact subset K. Let $\{\gamma_n\}$ be a sequence of future-pointing geodesic segments in K such that $\gamma_n(0) \to p$ and $\gamma_n(1) \to q \neq p$. Then, there is a future-pointing causal broken geodesic γ from p to q and a subsequence $\{\gamma_m\}$ such that $\lim_{m\to\infty} L(\gamma_m) \leq L(\gamma)$.

Proof. Consider the limit sequence given by lemma 4.10. If it is infinite, the quasi-limit would be an inextendible timelike curve, so lemma 4.4 implies that it would eventually leave K, but no γ_n does it. Then, the limit sequence is finite, so quasi-limit is a broken geodesic from p to q. By the local length maximality in geodesics given by proposition 5.3, $L(\gamma_m|_{[s_{m,i},s_{m,i+1}]}) \leq L(p_i p_{i+1})$ and, joining them, we get the desired inequality. □

Proposition 5.6. If $\mathcal{M}$ is a globally hyperbolic spacetime, the time separation function τ is continuous in $\mathcal{M} \times \mathcal{M}$

Proof. By lemma 5.4, τ is lower semicontinuous. If it was not upper semicontinuous at $(p,q) \in \mathcal{M} \times \mathcal{M}$, there would exist $\delta > 0$ and sequences $\{p_n\} \to p$ and $\{q_n\} \to q$ such that $\tau(p_n, q_n) > \tau(p,q) + \delta$ for every $n \in \mathbb{N}$. By proposition 3.2, there exists some $p^- \ll p$ and $q^+ \gg q$ such that $p_n \in I^+(p^-)$ and $q_n \in I^-(q^+)$ for infinitely many n.

Since $\tau(p_n, q_n) > 0$, there exists a timelike curve α_n from p_n to q_n such that $L(\alpha_n) > \tau(p_n, q_n) - \frac{1}{n}$. According to theorem 4.13, $J(p^-, q^+)$ is compact so, by the preceding lemma, there exists a broken geodesic, γ, from p to q and a subsequence $\{\alpha_m\}$ of $\{\alpha_n\}$ such that $\lim_{m\to\infty} L(\alpha_m) \leq L(\gamma)$. Therefore,

$$\tau(p,q) \geq L(\gamma) \geq \lim_{m\to\infty} L(\alpha_m) \geq \lim_{m\to\infty} \left(\tau(p_m, q_m) - \frac{1}{m}\right) \geq \tau(p,q) + \delta$$

This contradiction proves that $\tau(p,q)$ is continuous when $\mathcal{M}$ is globally hyperbolic. □

6 Conjugate and focal points

Definition 6.1. Let γ be a geodesic in a manifold $\mathcal{M}$ whose tangent is ξ^a. A vector field η^a along γ is called a *Jacobi field* if it satisfies the Jacobi equation

$$\xi^a \nabla_a (\xi^b \nabla_b \eta^c) = -R_{abd}{}^c \eta^b \xi^a \xi^d$$

Jacobi fields are interesting because they generate a one-parameter variation whose longitudinal curves are geodesics (see [3] for further discussion).

Definition 6.2. If there is a geodesic γ joining $p, q \in \mathcal{M}$, p is said to be *conjugate* to q along γ if there is a non-zero Jacobi field vanishing at p and q.

Proposition 6.3. Let γ be a causal curve joining two points $p, q \in \mathcal{M}$. If γ maximizes proper length between p and q, then it is is a geodesic with no interior conjugate points of p.

Proof. Clearly γ has no loops. Then, given any point r in γ, let N be a simple neighbourhood of r. Given points a and b lying on the two connected components of $(\gamma \cap N) \setminus \{r\}$, the segment of γ between a and b must be a geodesic because, otherwise, proposition 5.3 would construct a longer curve. As being a geodesic is a local property, γ must be a geodesic itself.

A rigorous proof of the fact that geodesic with interior conjugate points do not minimize proper length could be seen in [4]. However, an intuitive argument follows from the fact that there is a one-parameter variation of geodesics such that all of them contain both p and its conjugate point, c. If γ was a minimizing geodesic from p to q, it would be possible to construct broken geodesic of the same length that went from p to its conjugate point along another geodesic in that variation different from γ and from c to q by the segment of γ. This new geodesic could not be maximal because it would fail to be a geodesic at c, contrary to the first part of this proof. □

Definition 6.4. Given a submanifold Σ of a spacetime $\mathcal{M}$, a point p along a geodesic γ which intersects Σ orthogonally is called a *focal point* if there is a Jacobi field along γ that vanishes at p and generates a family of geodesics that intersect Σ orthogonally.

Lemma 6.5. Let $\mathcal{M}$ be a globally hyperbolic spacetime and let $p, q \in \mathcal{M}$ be such that $p < q$. Then, there is a causal geodesic γ from p to q such that $L(\gamma) = \tau(p, q)$.

Proof. By definition 5.2, there is a sequence $\{\gamma_n\}$ of causal curves from p to q whose lengths converge to $\tau(p, q)$. Then, by theorem 4.13 and lemma 5.5, there is a broken geodesic γ such that $L(\gamma) = \tau(p, q)$. Because γ is a maximal curve, proposition 6.3 implies that γ is a geodesic. □

Lemma 6.6. Let S be an achronal set. If $p \in \text{int}(D(S)) \setminus I^-(S)$, then $K := J^-(p) \cap D^+(S)$ is compact.

Proof. If $p \in S$, then $K = \{p\}$, so it is compact. Otherwise, $p \in \text{int}(D(S)) \cap I^+(S)$. Then let $\{x_n\}$ be a sequence in $J^-(p) \cap D^+(S)$ and let $\gamma_n \subset J^-(p) \cap D^+(S)$ be past-pointing causal curves from p to x_n. If p is an accumulation point of $\{x_n\}$, there is nothing to prove. Otherwise, lemma 4.10 guarantees that there is a limit sequence $\{p_i\}$ starting at p. If it is infinite, the corresponding quasi-limit given in definition 4.11 is inextendible, so it intersects $I^-(S)$, which is imposible because no γ_n does that. If $\{p_i\}$ is finite, there is a subsequence $\{x_m\}$ converging to some $x \in J^-(p)$. If $x \notin D^+(S)$, then $x \in I^-(S)$, so there is $x_N \in I^-(S)$, what contradicts the achronality of S. □

Theorem 6.7. Let S be a closed, achronal, edgeless, spacelike hypersurface in $\mathcal{M}$. Given $q \in$ int $(D^+(S))$, there is a future-pointing causal geodesic γ from S to q whose length is $\tau(S, q)$. Hence, γ is normal to S and has no focal points between S and q.

Proof. Considering $\text{int}(D(S))$ as a whole spacetime, S is a Cauchy surface, so $\text{int}(D(S))$ is globally hyperbolic. By lemma 6.6, $J^-(q) \cap D^+(S)$ is compact, so $J^-(q) \cap S$ is also compact because S was closed. By lemma 5.6, $\tau(x, q)$ is continuous, so it takes a maximum in the compact set $J^-(q) \cap S$. Hence, there exists $p \in S$ such that $\tau(p, q) = \tau(S, q)$. Furthermore, lemma 6.5 implies the existence of a geodesic γ from p to q such that $L(\gamma) = \tau(p, q) = \tau(S, q)$, which will be normal to S and will have no focal points before q. □

Theorem 6.8. Let $\mathcal{M}$ be a globally hyperbolic spacetime and let $S \subset \mathcal{M}$ be an achronal spacelike surface. Then, every point $q \in E^+(S) := J^+(S) \setminus I^+(S)$ lies on a future-pointing null geodesic orthogonal to S with no focal points between S and q.

Proof. If $q \in E^+(S)$, then $q \in J^+(p) \setminus I^+(p)$ for some $p \in S$. By corollary 3.6, q lies on a null geodesic from p to q. If this geodesic γ was not orthogonal to S or if it had a focal point before q, γ would not maximize proper length from S to q, so there will be a timelike curve from S to q, contrary to the fact that $q \notin I^+(S)$. □

7 Raychaudhuri equation and energy conditions

7.1 Timelike geodesics

For deriving the Raychaudhuri equation, we start from the definition of the Riemann curvature tensor

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) u^\alpha = R^\alpha{}_{\beta\mu\nu} u^\beta$$

Then, contracting α with μ and multiplying by u^ν, we get

$$u^\nu \nabla_\mu \nabla_\nu u^\mu - u^\nu \nabla_\nu \nabla_\mu u^\mu = R_{\beta\nu} u^\beta u^\nu$$

where $R_{\beta\nu}$ is the Ricci tensor. Using Leibniz rule for the covariant derivative,

$$u^\nu \nabla_\nu \nabla_\mu u^\mu + \nabla_\mu u_\nu \nabla^\nu u^\mu - \nabla_\mu \left(u^\nu \nabla_\nu u^\mu \right) + R_{\beta\nu} u^\beta u^\nu = 0 \tag{1}$$

Let's now suppose that u^μ represents an affinely parametrised timelike geodesic vector field normal to a spacelike hypersurface, which means that $u^\nu \nabla_\nu u^\mu = 0$. Therefore, the third term in the preceding equation vanishes. Defining the spatial metric as $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$ and the *expansion* θ, *shear* $\sigma_{\mu\nu}$ and *twist* $\omega_{\mu\nu}$ as follows:

$$\begin{aligned} \theta &:= h_{\mu\nu} \nabla^\nu u^\mu \\ \sigma_{\mu\nu} &:= \nabla_{(\mu} u_{\nu)} - \frac{1}{3} \theta h_{\mu\nu} \\ \omega_{\mu\nu} &:= \nabla_{[\nu} u_{\mu]} \end{aligned}$$

Then, Raychaudhuri equation 1 reads

$$u^\rho \nabla_\rho \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu$$

Parametrising geodesics by their proper time, $u^\mu = \nabla^\mu \tau$, so twist $\omega_{\mu\nu}$ vanishes. On the other hand, $\sigma_{\mu\nu}\sigma^{\mu\nu}$ is clearly non-negative because it is a 'purely spatial' tensor. Then, if the congruence condition $R_{\mu\nu} u^\mu u^\nu \geq 0$ holds for every timelike vector, we get the folllowing inequality:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \Rightarrow \frac{d\theta^{-1}}{d\tau} \geq \frac{1}{3} \Rightarrow \theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau$$

where θ_0 is the initial value of the expansion. If θ_0 is negative, the preceding equation affirms that θ reach value $-\infty$ within a proper time less than $3/|\theta_0|$. Next proposition follows from this fact.

Proposition 7.1. Let $\mathcal{M}$ be an orientable spacetime satisfying that $R_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$ for every timelike vector field ξ^α. If Σ is a spacelike hypersurface such that $\theta(p) < 0$ at some point $p \in \Sigma$. Then, within proper time $\tau \leq 3/|\theta(p)|$ along the future-pointing geodesic γ orthogonal to Σ at p, there is a focal point of Σ, provided that γ could be extended that far.

7.2 Null geodesics

We are interested in finding a Raychaudhuri equation for null geodesics too. However, the main problem arises from the fact that every null vector belongs to its orthogonal space. Thus, every hypersurface orthogonal to a null geodesic has to contain it. Then, we will work with 2-dimensional spacelike surfaces orthogonal to a null geodesic γ. Let Σ be one of these surfaces. At each point $p \in \Sigma$, there are two future-pointing null tangent vectors orthogonal to Σ and, locally, it is possible to make a continuous designation of one of them.[8] Then, we have a set of null geodesics going "inwards" and another one going "outwards".[9]

By properties of the normal exponential map (see [4]) these sets of null geodesics describe locally tangent fields k_+^μ and k_-^μ. Nevertheless, there is no natural way of parametrising null geodesics because proper time cannot be defined. We will do that parametrization in an arbitrary

[8]Globally, it may not be possible to make such designation. This is the case of a Möbius strip.

[9]In analogy with the behaviour of null geodesics in a spherically symmetric metric.

way and just imposing next two conditions: the tangent field generated $k^\mu_\pm$ is smooth and there are scalar functions $u_\pm$ such that $k^\mu_\pm = \nabla^\mu u_\pm$. Again, we use the metric $h_{\mu\nu} = g_{\mu\nu} + T_\mu N_\nu + T_\nu N_\mu$, where T and N are two parallel transported null vectors orthogonal to Σ and normalised such that $N_\mu T^\mu = -1$. Again, *expansion* θ, *shear* $\sigma_{\mu\nu}$ and *twist* $\omega_{\mu\nu}$ are defined as follows:

$$\begin{aligned}\theta &:= h_{\mu\nu}\nabla^\nu u^\mu \\ \sigma_{\mu\nu} &:= \nabla_{(\mu}u_{\nu)} - \frac{1}{2}\theta h_{\mu\nu} \\ \omega_{\mu\nu} &:= \nabla_{[\nu}u_{\mu]}\end{aligned}$$

Raychaudhuri equation 1 is

$$k^\rho_\pm \nabla_\rho \theta_\pm = \frac{d\theta_\pm}{d\tau} = -\frac{1}{2}\theta^2_\pm - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu$$

Again, twist $\omega_{\mu\nu}$ vanishes because parametrisation has been chosen for the normal bundles $k^\mu_\pm$ being surface orthogonal and $\sigma_{\mu\nu}\sigma^{\mu\nu}$ is clearly non-negative since $h_{\mu\nu}$ is positive definite. If $R_{\alpha\beta}k^\alpha k^\beta \geq 0$ for every null vector k^μ, we get

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq 0 \Rightarrow \frac{d\theta^{-1}}{d\tau} \geq \frac{1}{2} \Rightarrow \theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{2}\tau$$

If the initial value of the expansion θ_0 is negative, the null geodesic expansion reach value $-\infty$ within the affine parameter interval $[0, 2/|\theta_0|]$. This fact is the reason why next proposition is true.

Proposition 7.2. Let $\mathcal{M}$ be a spacetime satisfying that $R_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$ for every null vector field ξ^α. Let S be a surface satisfying that the outgoing (resp. ingoing) expansion θ_+ (resp. θ_-) takes the negative value θ_0 at some point $q \in S$. Then, there is a focal point to S along the outgoing (resp. ingoing) null geodesic starting from q, within the affine parameter interval $[0, 2/|\theta_0|]$.

7.3 Energy conditions

Condition $R_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$ appearing in Raychaudhuri equation could be related to the stress tensor $T_{\alpha\beta}$ using Einstein equations. These equations could be written as [10]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Comparing traces of both sides, we get that $-R = 8\pi GT$, where T is the trace of the stress tensor. Then, we get that

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

If we evaluate this equality in a vector field ξ^μ which represents an affinely parametrised congruence of null geodesics, we get that

$$R_{\mu\nu}\xi^\mu\xi^\nu = 8\pi G T_{\mu\nu}\xi^\mu\xi^\nu$$

Then, Raychaudhuri condition $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ could be written as

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{2}$$

[10] Provided that there is no cosmological constant.

which is usually denoted by the *weak energy condition.* [11]

On the other hand, when ξ^μ represents a congruence of timelike geodesics parametrized by their proper time, Einstein equations could be written as

$$R_{\mu\nu}\xi^\mu\xi^\nu = 8\pi G\left(T_{\mu\nu}\xi^\mu\xi^\nu + \frac{1}{2}T\right)$$

so the congruence condition $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ becomes

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T \tag{3}$$

which is commonly called as the *strong energy condition.*[12]

It is interesting to understand what do these energy conditions mean to physically reasonable matter. For instance, consider an anisotropic fluid whose stress tensor is

$$T_{\mu\nu} = \rho t_\mu t_\nu + p_1 x_\mu x_\nu + p_2 y_\mu y_\nu + p_3 z_\mu z_\nu$$

where $\{t^\mu, x^\mu, y^\mu, z^\mu\}$ is an orthonormal basis, such that t^μ is timelike. In this situation, ρ could be understood as the *rest energy density of matter* and p_1, p_2 and p_3 are called *principal pressures.*[13]

In this case, the weak energy condition is equivalent to

$$\rho \geq 0 \ \ \& \ \ \rho + p_i \geq 0 \ \forall i = 1,2,3.$$

while the strong energy condition is equivalent to

$$\rho + p_1 + p_2 + p_3 \geq 0 \ \ \& \ \ \rho + p_i \geq 0 \ \forall i = 1,2,3.$$

Therefore, both weak and strong energy conditions are satisfied provided that $\rho \geq 0$ and there do not exist negative pressures. Thus, it is believed that all physically reasonable matter satisfy both conditions. However, dark energy, which could be responsible for the accelerated expansion of the Universe, may not satisfy the strong energy condition.

8 Singularity theorems for timelike geodesics

There are two main singularity theorems related to timelike geodesics. First theorem assumes more but proves more too.

Theorem 8.1. If a spacetime $\mathcal{M}$ satisfies that

1. $R_{\mu\nu}u^\mu u^\nu \geq 0$ for every timelike vector u^μ.
2. $\mathcal{M}$ contains a spacelike Cauchy hypersurface Σ with future expansion $\theta(p) \leq \theta_0 \leq 0 \ \forall p \in \Sigma$.

Then every future-pointing timelike curve starting from Σ has length at most $3/|\theta_0|$.

Proof. If $q \in I^+(\Sigma) = D^+(\Sigma) \setminus \Sigma$, by theorem 6.7 there is a timelike geodesic γ such than $L(\gamma) = \tau(S, q)$. Therefore, γ is a geodesic with no focal points along it between Σ and q and orthogonal to Σ at some p. However, proposition 7.1 asserts that there is a focal point along γ if its length is greater than $3/|\theta(p)| \leq 3/|\theta_0|$. Consequently, the distance to Σ from every point in $I^+(\Sigma)$ is at most $3/|\theta_0|$. By definition 5.2, every future-pointing timelike curve starting from S has length less than $3/|\theta_0|$. □

[11]Although we are describing them for null vectors, a spacetime is said to satisfy the energy conditions only if the inequality is true for every causal vector.

[12]There is no mathematical implication between weak and strong energy conditions. Names are just related to the fact that it appears physically more reasonable to assume the weak energy condition than the strong one.

[13]In particular, an anisotropic fluid is a perfect fluid whenever $p_1 = p_2 = p_3$.

However, being globally hyperbolic seems to be a strong condition which could be false in general spacetimes. For instance, in an everywhere expanding Universe, it may seem physically more reasonable to think it must fail to be globally hyperbolic instead of being singular. To avoid the global hyperbolicity assumption, the main price to pay is that the trapped hypersurface must be compact, so we are working with closed universes or with bounded regions of them, like black holes.

Theorem 8.2. If a strongly causal spacetime $\mathcal{M}$ satisfies that

1. $R_{\mu\nu}u^\mu u^\nu \geq 0$ for every timelike vector u^μ.
2. $\mathcal{M}$ contains a compact, edgeless, achronal, smooth spacelike hypersurface S with future convergence $\theta(p) \leq \theta_0 \leq 0$.

Then, there is at least one future inextendible future directed timelike geodesic starting in S whose length is no greater than $3/|\theta_0|$.

Proof. Suppose, for the sake of contradiction, that every future-pointing inextendible timelike geodesic has length greater than $3/|\theta_0|$. Since the open set $\text{int}\,(D(S)) \subset \mathcal{M}$, considered as a whole spacetime, satisfies the hypothesis of theorem 8.1, then every geodesic from S must leave $\text{int}\,(D(S))$ and lemma 4.15 implies that it intersects $H^+(S)$ before its length becomes greater than $3/|\theta_0|$. In particular, $H^+(S) \neq \emptyset$.

In the normal bundle of S, consider the set B formed by all zero vectors and all future-pointing causal vectors v satisfying that $-||v|| \leq 3/|\theta_0|$. Clearly, B is compact, since S is compact too. If $q \in H^+(S)$, then lemma 4.15 implies that there exists a sequence $\{q_n\} \subset D^+(S)$ converging to q. For each q_n, theorem 6.7 guarantees the existence of $\nu_n \in B$ such that $\exp(\nu_n) = q_n$. By the compactness of B, a subsequence of $\{\nu_n\}$ converges to some $\nu \in B$. Since, by construction, $-||\nu_n|| = \tau(S, q_n)$ and, by proposition 5.4, τ is lower continuous, so $-||\nu|| \geq \tau(S, q)$. Since we are supposing that every inextendible geodesic has length greater than $3/|\theta_0| \geq -||v||$, the geodesic γ_ν, whose starting point is p and initial tangent vector is ν, is defined in $[0, 1]$, so $q = \exp(\nu)$, which means that $\tau(S, q) \geq -||\nu||$. Hence, $H^+(S) \subset \exp(B)$.

By the assumption made about the lengths of the geodesics, the normal exponential map is defined on the whole B, so $H^+(S)$ is compact since it is closed and contained in the continuous image of B. However, since $edge(S) = \emptyset$, theorem 4.16 implies the existence of a future inextendible null geodesic lying entirely in $H^+(S)$. Since $\mathcal{M}$ is strongly causal, this fact contradicts lemma 4.4. This contradiction proves the existence of a future directed inextendible geodesic whose length is less than $3/|\theta_0|$. □

9 Singularity theorems for null geodesics

These singularity theorems are related to the existence of incomplete null geodesics. They are commonly used in a context related to gravitational collapse and black holes. When working with null geodesics, the main problem that arises is they cannot be orthogonal to spacelike hypesurfaces. In fact, if a null geodesic is orthogonal to a spacelike submanifold, its codimension has to be greater or equal than 2.

Definition 9.1. A spacelike surface S of a spacetime $\mathcal{M}$ is said to be *future-converging* if both of its families of normal null geodesics have negative future expansion.

Definition 9.2. A closed achronal set S is called *future-trapped* provided that $E^+(S) := J^+(S) \setminus I^+(S)$ is compact.

Theorem 9.3. If a space-time $\mathcal{M}$ satisfies that:

1. $R_{\mu\nu}u^\mu u^\nu \geq 0$ for every null vector u^μ.

2. $\mathcal{M}$ contains a non-compact, connected Cauchy hypersurface Σ.

3. $\mathcal{M}$ contains a compact surface S whose future expansion satisfies that $\theta_\pm(p) \leq \theta_0 \leq 0\ \forall p \in S$.

Then $\mathcal{M}$ is future null incomplete.

Proof. Let's suppose, for the sake of contradiction, that $\mathcal{M}$ is future null complete.

Globally, it may not be possible to construct a normal null vector field of S that does not vanish anywhere. However, locally are there two linearly independent, orthogonal, null vectors at any point $p \in S$. These null normal vectors $\tilde{S}$ constitute a double covering of S. Hence $\tilde{S}$ is compact.

By proposition 7.2 and theorem 6.8, $E^+(S) \subset \exp(K)$, where $K = \{\alpha v : v \in \tilde{S}, \alpha \in [0, 2/|\theta_0|]\}$. Since $\tilde{S}$ is compact, K is compact and so it is $\exp(K)$. Then, given a sequence $\{q_n\} \subset E^+(S)$, it has an accumulation point $q \in \exp(K) \subset J^+(S)$. Nevertheless, $q \notin I^+(S)$ since no q_n does, so $q \in E^+(S)$. Thus $E^+(S)$ is compact, so S is future-trapped.

Because $\mathcal{M}$ is time-orientable, it is posible to find a smooth timelike vector field ξ^μ (see [6] for further discussion). Since $E^+(S)$ is clearly achronal, every integral curve can intersect $E^+(S)$ at most once, while it intersects Σ exactly once, because of the definition of Cauchy surface. Then, the integral curves define a one-to-one map $\psi : E^+(S) \to \Sigma$ where $f(p)$ is given by the unique point in Σ which is in the same integral curve than p. It is proven in [4] that ψ is continuous. By the invariance of domain theorem (see [11]), $\psi(E^+(S))$ is an open subset or Σ. Furthermore, $\psi(E^+(S))$ is compact, so it is closed in Σ. As Σ is connected, $\psi\left(E^+(S)\right) = \Sigma$, which contradicts the fact that Σ was non-compact. This contradiction proves that our assumption was wrong, so $\mathcal{M}$ is future null incomplete. In particular, there exists a future-inextendible null geodesic starting from S whose length is less than $2/|\theta_0|$.

□

Again, global hyperbolicity is an unwanted hypothesis. Nevertheless, it can be eliminated by adding further assumptions. Then, we expose next theorem whose proof can be seen in [5].

Theorem 9.4. If a spacetime $\mathcal{M}$ satisfies that:

1. $R_{\mu\nu}u^\mu u^\nu \geq 0$ for every null vector u^μ.

2. Each timelike or null geodesic γ has a point such that $u_{[\rho}R_{\alpha]\beta\lambda[\mu}u_{\sigma]}u^\beta u^\lambda \neq 0$.

3. There exists no closed causal curve.

4. At least one of the following properties hold:

   (a) $\mathcal{M}$ contains a compact achronal set without edge.
   (b) $\mathcal{M}$ contains a future-trapped surface.
   (c) There is a point $p \in \mathcal{M}$ such that the expansion of the future directed null geodesics emanating from p becomes negative along each geodesic in the congruence.

Then, $\mathcal{M}$ contains at least one incomplete timelike or null geodesic.

Second hypothesis, which is called *generic condition*, means that the tidal force felt by causal geodesics will not be everywhere aligned with their tangent vector. It is assumed to avoid pathological spacetimes.

10 Singularities in some spacetimes

10.1 Singularity theorems in Friedmann-Lemâitre-Robertson-Walker metric

Friedmann-Lemâitre-Robertson-Walker metric model the evolution of the Universe and its line element could be written as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

where k is a real parameter which represents the curvature of the spatial sections and $a(t)$ is a function, called *scale factor*, which takes only positive values and whose derivative determines if the Universe is expanding or contracting.

If the Universe is dominated by matter or radiation, the stress tensor takes the form $T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}$, where ρ is the density of energy and p is the pressure. In case that matter is dominating, $p = 0$, while $p = \frac{\rho}{3}$ when the Universe is dominated by radiation. Therefore, both weak and strong energy conditions are satisfied whenever the Universe is dominated by matter or radiation.

In this case, the hypersurface specified by $t = t_0$ is a Cauchy surface whose past expansion is $\theta = -3\frac{\dot{a}}{a} = -3H$, where H is the Hubble parameter. If the Universe is expanding, then $H > 0$ and theorem 8.1 implies that there is a past singularity, commonly called Big-Bang. In particular, it establishes a bound about the age of the Universe: $t \leq H^{-1}$.

However, observational advances have discovered that the Universe expansion may be ruled by the presence of dark energy, which might have negative pressures and may not satisfy the strong energy condition. Then, singularity theorems could not be used, so the Big Bang could be avoided. Nevertheless, there are other experimental evidences that confirm the existence of a Big Bang, like microwave cosmic background observations.

10.2 Singularity theorems in Schwarzschild metric

Schwarzschild spacetime is the only metric modelling an empty spherically symmetric spacetime. In particular, it could describe a non-rotating black hole. Its line element could be written as

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where m is a real parameter which represents the black hole mass.

There is a problem when we reach the Schwarzschild radius $r = 2m$ from infinity. Beyond this point, radial coordinate becomes timelike while time coordinate becomes spacelike. Therefore, we cannot define a continuous designation about which time direction is the future one, so spacetime is not time-orientable. This fact, combined with coordinate r blowing up at Schwarzschild radius, makes it difficult to work with this set of coordinates

Instead of that, we will use Kruskal coordinates for expanding this metric inside Schwarzschild radius. Using the change of coordinates given by

$$T = \frac{1}{2}\exp\left(\frac{1}{4m}\left(r + 2m\ln\left(\frac{r}{2m} - 1\right) + t\right)\right) - \frac{1}{2}\exp\left(\frac{1}{4m}\left(r + 2m\ln\left(\frac{r}{2m} - 1\right) - t\right)\right)$$

$$X = \frac{1}{2}\exp\left(\frac{1}{4m}\left(r + 2m\ln\left(\frac{r}{2m} - 1\right) + t\right)\right) + \frac{1}{2}\exp\left(\frac{1}{4m}\left(r + 2m\ln\left(\frac{r}{2m} - 1\right) - t\right)\right)$$

the metric becomes

$$ds^2 = \frac{32m^3 e^{-r/2m}}{r}(-dT^2 + dX^2) + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where $r(T, X)$ is the inverse coordinate transformation, given by the equation

$$\left(\frac{r}{2m} - 1\right) e^{r/2m} = X^2 - T^2$$

.

In this transformation, the external region outside the Schwarzschild black hole, $r > 2m$, transforms isometrically to $A_1 = \{(T, X) \in \mathbb{R}^2 : X^2 > T^2, X > 0\}$ and does the same to $A_2 = \{(T, X) \in \mathbb{R}^2 : X^2 > T^2, X < 0\}$. The internal area maps isometrically to $B_1 = \{(T, X) \in \mathbb{R}^2 : 0 \leq T^2 - X^2 \leq 1, T > 0\}$ and to $B_2 = \{(T, X) \in \mathbb{R}^2 : 0 \leq T^2 - X^2 \leq 1, T < 0\}$. [14]

Without loss of generality, we can suppose that the part of the Universe outside the black hole is identified with A_1. Then, the extension of the metric could be made by adding B_1 or B_2.[15] In the first case, we have a black hole, where no causal future-pointing geodesic can leave it, while the second case is a white hole and no causal future directed geodesic can enter in it. As white holes have not been observed in Nature yet, we will just consider the black hole extension.

Kruskal extension is conformal to Minkowski metric, so future timelike vectors are the same in both spacetimes. Then, it is easily seen in figure 1 that every causal future directed path starting inside Schwarzschild radius, in the black hole extension, will eventually reach the singularity $r = 0$.

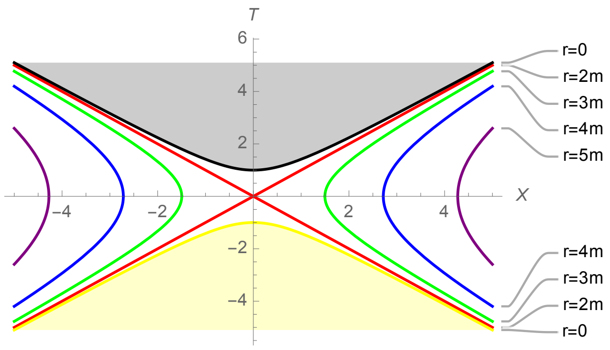

Figure 1: Graphical representation of Kruskal extension for Schwarzschild black hole. Singularity inside the black hole is coloured in black and the one inside white hole is coloured in yellow.

We can check that this spacetime verifies the hypothesis of theorem 9.3. First of all, Ricci tensor vanishes, so the congruence condition is verified trivially. Moreover, this spacetime is globally hyperbolic, as $\{T = 0\}$ is clearly a Cauchy surface.[16] Finally, considering spheres whose radius are less than Schwarzschild one, they are represented in Kruskal extension by fixing T and X. Then, orthogonal future-pointing null vectors are $(1, \pm 1, 0, 0)$. Then, future expansion is $\theta_\pm = \frac{4m}{r^2} e^{-r/2m}(r - 2m)(T \mp X) < 0$. Then, the spheres inside black holes are future-converging surfaces.

We knew that no null geodesic could leave the black hole and all of them would eventually reach the singularity, but now theorem 9.3 asserts that al least one of them reaches the singularity within a finite amount of proper time, so spacetime is singular.

[14] In Schwarzschild metric, no causal path could cross Schwarzschild radius without spending an infinite amount of time. Nevertheless, time is just a coordinate whose only physical meaning is the time measured by an observer placed at infinity. However, geodesics approaching to Schwarzschild radius could reach it in a finite amount of proper time (although t becomes infinity) and then they will continue their travel inside the black hole.

[15] Note that one is equivalent to the other but interchanging future and past.

[16] We can consider the whole extension to $\mathbb{R}^2$, i.e. the Universe is duplicated. Nevertheless, it is not an inconvenient as geodesics neither can leave the black hole nor intersects both copies of the Universe.

11 Concluding statement

The existence of singularities is the main theoretical limitation to General Relativity on its purpose of explaining the physical world. In this work, we have proven that there are incomplete geodesics in globally hyperbolic spacetimes containing trapped surfaces and satisfying the congruence conditions. This the case of an expanding Universe modelled by Friedmann-Lemâitre-Robertson-Walker metric and a non-rotating black hole.

The strength of the singularity theorems comes from the fact that they do not need to assume any kind of symmetry for proving that the spacetime is singular. Therefore, a hypothetical breaking in the spherical symmetry of a gravitational collapse that forms a black hole could not avoid the existence of a singularity.

There are two alternatives to afford the problem that arises from the existence of singularities predicted by these theorems. On the one hand, the main proposal is to try to find solutions for the Einstein field equations that avoid some assumptions of the theorems and, consequently, the existence of singularities. For instance, it is possible to avoid the congruence condition including a cosmological constant in our theory or one may try to find non-globally hyperbolic solutions to Einstein equations. On the other hand, existence of singularities could be assumed and one could try to analyse how will physical laws are modified in this context. Nowadays, an undiscovered quantum gravity theory seems to be the most the promising approach.

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